VARIABLE ORDER MIXED h-FINITE ELEMENT METHOD FOR LINEAR ELASTICITY WITH WEAKLY IMPOSED SYMMETRY. II. AFFINE AND CURVILINEAR ELEMENTS IN 2D

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ABSTRACT. We continue our study on variable order Arnold-Falk-Winther elements for 2D elasticity in context of both affine and parametric curvilinear elements. We present an h-stability result for affine elements, and an asymptotic stability result for curvilinear elements. Both theoretical results are confirmed with numerical experiments.

1. INTRODUCTION

1.1. **Elasticity problem.** Linear elasticity is a classical subject, and it has been studied for a long time. The presented research is motivated with a class of time-harmonic problems formulated as follows. Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, denote a bounded domain occupied by the elastic body. Assume that the boundary of Ω, $\Gamma = \partial\Omega$ has been split into two disjoint subsets Γ_1 and Γ_2,

$$\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2}, \quad \Gamma_1 \cap \Gamma_2 = \emptyset$$

Subsets Γ_1, Γ_2 are assumed to be (relatively) open in Γ. We seek:

- displacement vector $u_i(x)$, $x \in \Omega$,
- linearized strain tensor $\epsilon_{ij}(x)$, $x \in \Omega$, and
- stress tensor $\sigma_{ij}(x)$, $x \in \Omega$

that satisfy the following system of equations and boundary conditions.

- Cauchy's geometrical relation between the displacement and strain,

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad x \in \Omega \tag{1.1}$$

- Equations of motion resulting from the principle of linear momentum,

$$-\sigma_{ij,j} - \rho(x)\omega^2 u_i = f_i(x), \quad x \in \Omega \tag{1.2}$$

- Symmetry of the stress tensor being a consequence of the principle of angular momentum,

$$\sigma_{ij} = \sigma_{ji}, \quad x \in \Omega$$

- Constitutive equations for linear elasticity,

$$\sigma_{ij} = E_{ijkl}(x)\epsilon_{kl}, \quad x \in \Omega$$

- Kinematic boundary conditions,

$$u_i = u_i^0, \quad x \in \Gamma_1$$

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- Traction boundary conditions,

$$\sigma_{ij}n_j = g_i(x), \quad x \in \Gamma_2$$

Here:

- ρ is the density of the body,
- f_i are volume forces prescribed within the body,
- g_i are tractions prescribed on Γ_2 part of the boundary,
- u_i^0 are displacements prescribed on Γ_1 part of the boundary,
- n_j is the unit outward normal vector for boundary Γ, and
- E_{ijkl} is the tensor of elasticities.

As usual, commas denote partial derivatives, and we use the Einstein's summation convention. Vector $t_i := \sigma_{ij}n_j$ is known as the traction or stress vector.

The geometrical relations imply the symmetry of the strain tensor. The symmetry of strain and stress tensors imply then the usual (minor) symmetry conditions for the elasticities,

$$E_{ijkl} = E_{jikl} = E_{ijlk} \tag{1.3}$$

The laws of thermodynamics imply an additional (major) symmetry condition,

$$E_{ijkl} = E_{klij}$$

and positive definiteness condition,

$$E_{ijkl}\xi_{ij}\xi_{kl} > 0, \quad \forall \xi_{ij} = \xi_{ij}$$

The last condition implies that the constitutive equation can be inverted to represent strains in terms of stresses,

$$\epsilon_{kl} = C_{klij}\sigma_{ij} \tag{1.4}$$

where $C_{klij} = E^{-1}_{ijkl}$ is known as the compliance tensor and satisfies analogous symmetry and positive-definiteness properties.

For isotropic materials, the elasticities tensor is expressed in terms of two independent Lame's constants μ and λ,

$$E_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}$$

and the constitutive equation reduces to,

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}$$

where δ_{ij} denotes the Kronecker's delta. Inverting, we obtain

$$\epsilon_{ij} = A\sigma_{ij}, \quad \epsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(2\mu + n\lambda)}\sigma_{kk}\delta_{ij}$$

Of particular interest is the case of nearly incompressible material corresponding to $\lambda \to \infty$. Notice that the norm of the elasticities blows up then to infinity but the norm of the compliance tensor remains uniformly bounded. This suggests that formulations based on the compliance relation have a chance to remain uniformly stable for nearly incompressible materials. Another reason of using formulations based on compliance relation is for (visco)elastic vibration problems for structures with large material contrast. For a more detailed discussion on these problems, we refer to the introduction in [19].

1.2. **Variational formulations for elasticity.**

Dual-Mixed Formulation. We eliminate the strain tensor and combine geometrical relations (1.1) and constitutive equation (1.4) in a single equation,

$$C_{klij}\sigma_{ij} = \frac{1}{2}(u_{k,l} + u_{l,k})$$

Classical formulation for elasticity is based on satisfying the equation above in a strong sense while relaxing the momentum equations. The idea behind the dual-mixed formulation for elasticity is exactly opposite, the equation above is relaxed whereas momentum equations (1.2) are satisfied in a strong sense.

The final formulation includes building in kinematic boundary conditions and it reads as follows.

$$\left\{\begin{aligned} &\sigma \in H(\mathrm{div}, \Omega, \mathbb{S}) \,:\, \sigma_{ij}n_j &&= g_i \text{ on } \Gamma_2,\ u \in L^2(\Omega, \mathbb{V}) \\ &\int_\Omega C_{ijkl}\sigma_{kl}\tau_{ij} + \int_\Omega u_i\tau_{ij,j} &&= \int_{\Gamma_1} u_i^0 \tau_{ij}n_j \quad \forall \tau \in H(\mathrm{div}, \Omega, \mathbb{S}) \,:\, \tau_{ij}n_j = 0 \text{ on } \Gamma_2 \\ &-\int_\Omega \sigma_{ij,j}v_i - \omega^2 \int_\Omega \rho u_i v_i &&= \int_\Omega f_i v_i \quad \forall v \in L^2(\Omega, \mathbb{V}) \end{aligned}\right. \tag{1.5}$$

Above, $\mathbb{V} = \mathbb{R}^3$ and $\mathbb{S}$ denotes the space of symmetric tensor. $L^2(\Omega, \mathbb{V})$ denotes the space of square integrable functions with values in $\mathbb{V}$, and $H(\mathrm{div}, \Omega, \mathbb{S})$ denotes the space of square-integrable functions with values in $\mathbb{S}$, whose row-wise divergence lives in $L^2(\Omega, \mathbb{V})$. The traction conditions are satisfied in the sense of traces for functions from $H(\mathrm{div}, \Omega)$ (they live in $H^{-1/2}(\Gamma)$).

The corresponding static case can be derived formally by considering the so-called Hellinger-Reissner functional, and the variational formulation is frequently identified as the Hellinger-Reissner variational principle.

Dual-Mixed Formulation with Weakly Imposed Symmetry. The symmetry condition is difficult to enforce on the discrete level. This has led to the idea of relaxing the symmetric function and satisfying it in a weaker, integral form. This is obtained by introducing tensor-valued test functions q with values in the space of *antisymmetric* tensors $\mathbb{K} := \{q_{ij} \,:\, q_{ij} = -q_{ji}\}$, and replacing the symmetry condition with an integral condition,

$$\int_\Omega \sigma_{ij}q_{ij} = 0, \quad \forall q \in L^2(\Omega, \mathbb{K})$$

On the continuous level, the integral condition implies the pointwise condition (understood in the L^2 sense), but on the discrete level, with an appropriate choice of spaces, the integral condition does not necessary imply the symmetry condition pointwise.

The extra condition leads to an extra unknown. The derivation of the weak form of the constitutive equation has to be revisited. We start by introducing the *tensor of infinitesimal rotations*,

$$p_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$$

Upon eliminating the strain tensor, the constitutive equation in the compliance form is now rewritten as,

$$C_{ijkl}\sigma_{kl} = \frac{1}{2}(u_{i,j} + u_{j,i}) = u_{i,j} - p_{ij}$$

Multiplication with a test function τ (now, not necessarily symmetric), integration over Ω, and integration by parts, leads to a new relaxed version of the equation,

$$\int_\Omega C_{ijkl}\sigma_{kl}\tau_{ij} = -\int_\Omega u_i\tau_{ij,j} + \int_\Gamma u_i\tau_{ij}n_j - \int_\Omega p_{ij}\tau_{ij}$$

We obtain a new variational formulation in the form:
(1.6)

$$\begin{cases} \sigma \in H(\mathrm{div},\Omega,\mathbb{M}) \,:\, \sigma_{ij}n_j = g_i \text{ on } \Gamma_2,\ u \in L^2(\Omega,\mathbb{V}),\ p \in L^2(\Omega,\mathbb{K}) \\ \displaystyle\int_\Omega C_{ijkl}\sigma_{kl}\tau_{ij} + u_i\tau_{ij,j} + p_{ij}\tau_{ij} = \int_{\Gamma_1} u_i^0\tau_{ij}n_j \quad \forall\tau \in H(\mathrm{div},\Omega,\mathbb{M}) : \tau_{ij}n_j = 0 \text{ on } \Gamma_2 \\ \displaystyle -\int_\Omega \sigma_{ij,j}v_i - \omega^2\int_\Omega \rho u_i v_i = \int_\Omega f_i v_i \quad \forall v \in L^2(\Omega,\mathbb{V}) \\ \displaystyle\int_\Omega \sigma_{ij}q_{ij} = 0 \quad \forall q \in L^2(\Omega,\mathbb{K}) \end{cases}$$

In the above, $H(\mathrm{div},\Omega,\mathbb{M})$ denotes the space of tensor-valued square integrable fields with a square integrable divergence. The formulation for the static case can again be derived formally by looking for a stationary point of the so-called *Generalized Hellinger-Reissner* functional, and it is frequently identified as the *Generalized Hellinger-Reissner Variational Principle*.

In the analysis presented in this paper, we consider the static case only and, for the sake of simplicity we assume that the body is fixed on the entire boundary $\partial\Omega$, i.e. $\Gamma_2 = \emptyset$.

To connect with the notations used in [6],[2], we denote by A the compliance operator.

$$A : \mathbb{M} \ni \varepsilon_{ij} \longrightarrow \sigma_{ij} = C_{ijkl}\varepsilon_{kl} \in \mathbb{M}.$$

The operator A maps tensors to tensors, is bounded, symmetric and, for piecewise constant material properties, uniformly positive definite. Symmetry properties (1.3) imply that A maps $\mathbb{S}$ into itself. In the isotropic case, A has the form

$$A\sigma = \frac{1}{2\mu}\left(\sigma - \frac{\lambda}{2\mu + n\lambda}Tr\,(\sigma)\,I\right),$$

where $\lambda\,(x)\,,\mu\,(x)$ are the Lamé coefficients.

We can rewrite then formulation (1.5) with a more compact, index free notation.

Find $\sigma \in H\,(\mathrm{div},\Omega;\mathbb{S})$, and $u \in L^2\,(\Omega;\mathbb{V})$, satisfying

$$\int_\Omega (A\sigma : \tau + \mathrm{div}\tau \cdot u)\ dx = 0, \quad \tau \in H\,(\mathrm{div},\Omega;\mathbb{S})\,, \tag{1.7}$$
$$\int_\Omega \mathrm{div}\sigma \cdot v\, dx = \int_\Omega f \cdot v\, dx, \quad v \in L^2\,(\Omega;\mathbb{V})\,.$$

Similarly, the formulation (1.6) becomes to seek $\sigma \in H\left(\mathrm{div}, \Omega; \mathbb{M}\right)$, $u \in L^2\left(\Omega; \mathbb{V}\right)$, and $p \in L^2\left(\Omega; \mathbb{K}\right)$ satisfying

$$
\begin{aligned}
(1.8) \qquad & \int_\Omega \left(A\sigma : \tau + \mathrm{div}\tau \cdot u + \tau : p\right) \, dx = 0, \quad \tau \in H\left(\mathrm{div}, \Omega; \mathbb{M}\right), \\
& \int_\Omega \mathrm{div}\sigma \cdot v \, dx = \int_\Omega f \cdot v \, dx, \quad v \in L^2\left(\Omega; \mathbb{V}\right), \\
& \int_\Omega \sigma : q \, dx = 0, \quad q \in L^2\left(\Omega; \mathbb{K}\right).
\end{aligned}
$$

Dual-Mixed Formulation with Weakly Imposed Symmetry for Plane Elasticity. In two space dimensions, the skew-symmetric tensors involve a single non-zero component p,

$$
\begin{pmatrix} 0 & p \\ -p & 0 \end{pmatrix}
$$

The formulation (1.8) reduces to seek $\sigma \in H(\mathrm{div}, \Omega; \mathbb{M})$, $u \in L^2(\Omega; \mathbb{V})$, and $p \in L^2(\Omega)$ satisfying

$$
\begin{aligned}
(1.9) \qquad & \int_\Omega (A\sigma : \tau + \mathrm{div}\tau \cdot u - S_1\tau \, p) \, dx = 0, \quad \tau \in H(\mathrm{div}, \Omega; \mathbb{M}), \\
& \int_\Omega \mathrm{div}\sigma \cdot v \, dx = \int_\Omega f \cdot v \, dx, \quad v \in L^2(\Omega; \mathbb{V}), \\
& \int_\Omega S_1\sigma \, q \, dx = 0, \quad q \in L^2(\Omega).
\end{aligned}
$$

where operator S_1 maps a real 2×2 matrix to a real number. For any $\sigma \in \mathbb{R}^{2\times 2}$,

$$
S_1\sigma = \sigma_{12} - \sigma_{21}. \tag{1.10}
$$

1.3. **Review of the existing work.** Besides the Hellinger-Reissner and generalized Hellinger-Reissner principles discussed in the previous sections, there are other variational formulations for elasticity, see [18].

Stable finite element discretizations based on (1.7) are difficult to construct. A detailed description of related challenges can be founded in [14]. A stable discretization based on the strong enforcement of symmetry condition was developed in [2]. For the lowest order element, the number of degrees of freedom for stress tensor in 3D is 162.

Meanwhile, a number of authors have developed approximation schemes based on (1.8). See [5, 6, 14, 1, 3, 4, 15, 11, 17, 20, 21, 22, 23]. For a brief description of these methods, we refer to the introduction in [6]. Recently, Cockburn, Gopalakrishnan and Guzman have developed a new mixed method for linear elasticity using a hybridized formulation based on (1.8) in [10].

The work presented in this paper is based on mixed finite element methods developed by Arnold, Falk and Winther for the formulation (1.8) in [5, 14, 6]. We restrict our analysis to plane linear elasticity problem only.

At the first glance, a generalization to elements of variable order seems to be straightforward. And this is indeed the case from the implementation point of view. The stability analysis, however, is much more difficult. The canonical projection operators defined in [5] do not commute with exterior derivative operators for variable order elements, a property essential in the proof of discrete stability. In [19], we resolved the problem by invoking projection based (PB) interpolation

operators. Unfortunately, the PB operators do not commute with the operator S_{n-2}, invalidating construction of operator $\tilde{W}_h$ introduced in the stability analysis presented in [5] We resolved the problem in [19] by designing a new operator $\tilde{W}_h$. However, in our first contribution we were only able to prove the well-definedness of $\tilde{W}_h$ for $n = 2$ with orders varying from 0 to 3. In this paper, we propose PB operators of new type and a new operator $\tilde{W}_h$ that enable the h-stability analysis for 2D elements of variable order[1]. Additionally, we analyze curvilinear elements and establish conditions for their asymptotic h-stability.

1.4. **Scope of the paper.** An outline of the paper is as follows. Section 2 introduces notations. Section 3 reviews definitions of finite element spaces on affine meshes. In Section 4, we return to the mixed formulation for plane elasticity with weakly imposed symmetry, and recall Brezzi's conditions for stability. In Section 5, we establish necessary results for proving the stability for affine meshes. We design new PB operators and a new operator $\tilde{W}_h$. In Section 6 we prove that the Brezzi conditions are satisfied for affine meshes. The main result for affine meshes is Theorem 6.3. In Section 7, we discuss curvilinear meshes and geometry assumptions. Section 8 reviews definitions of finite element spaces on a reference triangle, on a (physical) curved triangle, and on curved meshes. In Section 9, we establish necessary results for proving the asymptotic stability for curvilinear meshes. This part of work is much more technical than that of Section 5 because of the curvilinear meshes. We explain briefly the substantial difference between the analysis for curvilinear meshes and that for affine meshes at the beginning of Section 9. In Section 10 we prove that the Brezzi conditions are satisfied asymptotically for curvilinear meshes, i.e. for meshes that are sufficiently fine. We conclude the paper with numerical experiments that illustrate and verify the presented theoretical results.

2. Notations

We denote by T an arbitrary triangle in $\mathbb{R}^2$. And let $\hat{T}$ be the reference triangle. We denote the set of subsimplexes of dimension k of T by $\triangle_k(T)$, $k = 0, 1, 2$, and the set of all subsimplexes of T by $\triangle(T)$. $\triangle_0(T)$ consists of all vertices of T, $\triangle_1(T)$ consists of all edges of T, and $\triangle_2(T) = \{T\}$.

For U, a bounded open subset in $\mathbb{R}^n$, we define:

$$C^k(\overline{U}) = \{u \in C^k(U) : D^\alpha \text{ is uniformly continuous on } U, \forall |\alpha| \leq k\}.$$

We also define

$$C^{k,1}(\overline{U}) = \{u \in C^k(\overline{U}) : D^\alpha \text{ is Lipschitz on } U, \forall |\alpha| = k\}.$$

$\mathbb{M}$ will denote the space of 2×2 real matrices. For any vector space $\mathbf{X}$, we denote by $L^2(\Omega; \mathbf{X})$ the space of square-integrable vector fields on Ω with values in $\mathbf{X}$. In the paper, $\mathbf{X}$ will be $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{M}$. When $\mathbf{X} = \mathbb{R}$, we will write $L^2(\Omega)$. The corresponding norms will be denoted with the same symbol $L^2(\Omega; \mathbb{M})$. The corresponding Sobolev space of order m, denoted $H^m(\Omega; \mathbf{X})$, is a subspace of $L^2(\Omega; \mathbf{X})$ consisting of functions with all partial derivatives of order less than or equal to m

[1]With an upper bound on the polynomial order.

in $L^2(\Omega;\mathbf{X})$. The corresponding norm will be denoted by $\|\cdot\|_{H^m(\Omega)}$. The space $H(\mathrm{div},\Omega;\mathbb{M})$ is defined by

$$H(\mathrm{div},\Omega;\mathbb{M}) = \{\sigma \in L^2(\Omega;\mathbb{M}) : \mathrm{div}\sigma \in L^2(\Omega;\mathbb{R}^2)\},$$

where divergence of a matrix field is the vector field obtained by applying operator div row-wise, i.e.,

$$\mathrm{div}\sigma = (\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2}, \quad \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2})^\top.$$

We introduce also a special space

$$H(\Omega) = \{\omega \in H(\mathrm{div},\Omega) : \omega|_{\partial\Omega} \in L^2(\partial\Omega;\mathbb{R}^2)\}$$

with the norm,

$$\|\omega\|_{H(\Omega)} = \|\omega\|_{H(\mathrm{div},\Omega)} + \|\omega\|_{L^2(\partial\Omega)}, \forall \omega \in H(\Omega).$$

For any scalar function u and any vector function ω with values in $\mathbb{R}^2$, we denote

$$\mathrm{curl}\ u = (\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1})^\top, \quad \mathrm{curl}\ \omega = \begin{bmatrix} \frac{\partial \omega_1}{\partial x_2}, & -\frac{\partial \omega_1}{\partial x_1} \\ \frac{\partial \omega_2}{\partial x_2}, & -\frac{\partial \omega_2}{\partial x_1} \end{bmatrix}.$$

Finally, by $\|\cdot\|$ we denote the standard 2-norm for vectors and matrices.

3. Finite element spaces

We begin by introducing the relevant finite element spaces on a single triangle T. Then, we define the finite element spaces on a whole affine triangular mesh $\mathcal{T}_h$ by "gluing" the finite element spaces on triangles.

3.1. Finite element spaces on a single triangle.

For any $r \in \mathbb{Z}_+ := \{n \in \mathbb{Z} : n \geq 0\}$, we introduce

$$\begin{aligned}
&\mathcal{P}_r(T) := \{\text{space of polynomials of order } r \text{ on } T\}, \\
&\mathcal{P}_r\Lambda^0(T) = \mathcal{P}_r\Lambda^2(T) := \mathcal{P}_r(T), \mathcal{P}_r\Lambda^1(T) := [\mathcal{P}_r(T)]^2, \\
&\mathring{\mathcal{P}}_r\Lambda^0(T) = \mathring{\mathcal{P}}_r(T) := \{w \in \mathcal{P}_r\Lambda^0(T) : w|_e = 0, \forall e \in \triangle_1(T)\}, \\
&\mathcal{P}_r^-\Lambda^1(T) := [\mathcal{P}_{r-1}(T)]^2 + (x_1, x_2)^\top \mathcal{P}_{r-1}(T), \\
&\mathcal{P}_r\Lambda^0(T;\mathbb{R}^2) = \mathcal{P}_r\Lambda^2(T;\mathbb{R}^2) := [\mathcal{P}_r(T)]^2, \\
&\mathcal{P}_r\Lambda^1(T;\mathbb{R}^2) := \left\{ \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} : (\sigma_{11}, \sigma_{12})^\top, (\sigma_{21}, \sigma_{22})^\top \in \mathcal{P}_r\Lambda^1(T) \right\}.
\end{aligned} \tag{3.1}$$

In [5, 14], spaces in (3.1) are defined in the language of exterior calculus. Here we just rewrite them using the language of calculus. We refer to [5] and [14] for a detailed correspondence before the exterior and classical calculus notations.

We denote by $\tilde{r}$ a mapping from $\triangle(T)$ to $\mathbb{Z}_+$ such that if $e, f \in \Delta(T)$ and $e \subset f$ then $\tilde{r}(e) \leq \tilde{r}(f)$. We introduce now formally the FE spaces of variable order.

Definition 3.1.

$$\mathcal{P}_{\tilde{r}}\Lambda^0(T) := \{u \in \mathcal{P}_{\tilde{r}(T)}\Lambda^0(T) : \forall e \in \triangle_1(T),\ u|_e \in \mathcal{P}_{\tilde{r}(e)}(e)\},$$

$$\mathcal{P}_{\tilde{r}}\Lambda^2(T) := \mathcal{P}_{\tilde{r}(T)}\Lambda^2(T) = \mathcal{P}_{\tilde{r}(T)}(T),$$

$$\mathcal{P}_{\tilde{r}}\Lambda^1(T) := \{\omega \in \mathcal{P}_{\tilde{r}(T)}\Lambda^1(T) : \forall e \in \triangle_1(T),\ \omega \cdot \mathbf{n}|_e \in \mathcal{P}_{\tilde{r}(e)}(e)\},$$

$$\mathcal{P}_{\tilde r}^{-}\Lambda^1(T) := \{\omega \in \mathcal{P}_{\tilde r(T)}^{-}\Lambda^1(T) : \forall e \in \triangle_1(T),\, \omega\cdot\mathbf{n}|_e \in \mathcal{P}_{\tilde r(e)-1}(e)\},$$
$$\mathcal{P}_{\tilde r}\Lambda^0(T;\mathbb{R}^2) := [\mathcal{P}_{\tilde r}\Lambda^0(T)]^2, \quad \mathcal{P}_{\tilde r}\Lambda^2(T;\mathbb{R}^2) := [\mathcal{P}_{\tilde r}\Lambda^2(T)]^2,$$
$$\mathcal{P}_{\tilde r}\Lambda^1(T;\mathbb{R}^2) := \left\{ \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} : (\sigma_{11},\sigma_{12})^\top, (\sigma_{21},\sigma_{22})^\top \in \mathcal{P}_{\tilde r}\Lambda^1(T) \right\}.$$

Here $\mathbf{n}$ is the outward normal unit vector along ∂T.

Remark 3.2. According to [5], for any $e \in \triangle_1(T)$, we have

$$\mathcal{P}_{\tilde r}\Lambda^0(T)|_e = \mathcal{P}_{\tilde r(e)}(e), \mathcal{P}_{\tilde r}\Lambda^1(T)|_e\cdot\mathbf{n} = \mathcal{P}_{\tilde r(e)}(e), \mathcal{P}_{\tilde r}^{-}\Lambda^1(T)|_e\cdot\mathbf{n} = \mathcal{P}_{\tilde r(e)-1}(e).$$

Actually, spaces in Definition 3.1 have been given in [19] with the language of exterior calculus.

3.2. **Finite element spaces on an affine triangular mesh.** Let $\mathcal{T}_h$ be an affine triangular mesh. We extend the $\tilde r$ mapping to a global map defined on $\triangle(\mathcal{T}_h)$ with values in $\mathbb{Z}_+$ such that if $e \subset f$, then $\tilde r(e) \le \tilde r(f)$.

Definition 3.3. We put $\Omega_h := \bigcup_{T\in\mathcal{T}_h} T$.

$$C\Lambda^0(\mathcal{T}_h) := \{u \in H^1(\Omega_h) : u \text{ is piece-wise smooth with respect to } \mathcal{T}_h\},$$
$$C\Lambda^1(\mathcal{T}_h) := \{\omega \in H(\mathrm{div},\Omega_h) : \omega \text{ is piece-wise smooth with respect to } \mathcal{T}_h\},$$
$$C\Lambda^2(\mathcal{T}_h) := \{u \in L^2(\Omega_h) : u \text{ is piece-wise smooth with respect to } \mathcal{T}_h\}.$$

We define

$$\mathcal{P}_{\tilde r}\Lambda^0(\mathcal{T}_h) := \{u \in C\Lambda^0(\mathcal{T}_h) : u|_T \in \mathcal{P}_{\tilde r}\Lambda^0(T), \forall T \in \mathcal{T}_h\},$$
$$\mathcal{P}_{\tilde r}\Lambda^1(\mathcal{T}_h) := \{\omega \in C\Lambda^1(\mathcal{T}_h) : \omega|_T \in \mathcal{P}_{\tilde r}\Lambda^1(T), \forall T \in \mathcal{T}_h\},$$
$$\mathcal{P}_{\tilde r}^{-}\Lambda^1(\mathcal{T}_h) := \{\omega \in C\Lambda^1(\mathcal{T}_h) : \omega|_T \in \mathcal{P}_{\tilde r}^{-}\Lambda^1(T), \forall T \in \mathcal{T}_h\},$$
$$\mathcal{P}_{\tilde r}\Lambda^2(\mathcal{T}_h) := \{u \in C\Lambda^2(\mathcal{T}_h) : u|_T \in \mathcal{P}_{\tilde r}\Lambda^2(T), \forall T \in \mathcal{T}_h\},$$
$$\mathcal{P}_{\tilde r}\Lambda^0(\mathcal{T}_h;\mathbb{R}^2) := [\mathcal{P}_{\tilde r}\Lambda^0(\mathcal{T}_h)]^2, \mathcal{P}_{\tilde r}\Lambda^2(\mathcal{T}_h;\mathbb{R}^2) := [\mathcal{P}_{\tilde r}\Lambda^2(\mathcal{T}_h)]^2,$$
$$\mathcal{P}_{\tilde r}\Lambda^1(\mathcal{T}_h;\mathbb{R}^2) := \left\{ \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} : (\sigma_{11},\sigma_{12})^\top, (\sigma_{21},\sigma_{22})^\top \in \mathcal{P}_{\tilde r}\Lambda^1(\mathcal{T}_h,\mathbb{R}^2) \right\}.$$

Remark 3.4. Spaces $\mathcal{P}_{\tilde r}\Lambda^0(\mathcal{T}_h), \mathcal{P}_{\tilde r}\Lambda^1(\mathcal{T}_h), \mathcal{P}_{\tilde r}^{-}\Lambda^1(\mathcal{T}_h), \mathcal{P}_{\tilde r}\Lambda^2(\mathcal{T}_h)$ coincide with those analyzed in [19].

Lemma 3.5.

$$\mathcal{P}_{\tilde r}\Lambda^1(\mathcal{T}_h) \subset \mathcal{P}_{\tilde r+1}^{-}\Lambda^1(\mathcal{T}_h) \subset \mathcal{P}_{\tilde r+1}\Lambda^1(\mathcal{T}_h),$$
$$div\mathcal{P}_{\tilde r+1}\Lambda^1(\mathcal{T}_h) \subset \mathcal{P}_{\tilde r}\Lambda^2(\mathcal{T}_h), curl\mathcal{P}_{\tilde r+1}\Lambda^0(\mathcal{T}_h) \subset \mathcal{P}_{\tilde r}\Lambda^1(\mathcal{T}_h).$$

Proof. This is straightforward. □

4. Mixed formulation for elasticity with weakly imposed symmetry

We assume that there is $r_{\max} \in \mathbb{N}$ such that, for any $h > 0$ and $f \in \Delta(\mathcal{T}_h)$, $\tilde r(f) \le r_{\max}$.

We recall the mixed formulation (1.8): Find $(\sigma, u, p) \in H(\mathrm{div},\Omega;\mathbb{M})\times L^2(\Omega;\mathbb{R}^2)\times L^2(\Omega)$ such that

$$\begin{aligned} \langle A\sigma,\tau\rangle + \langle \mathrm{div}\tau, u\rangle - \langle S_1\tau, p\rangle &= 0, \quad \tau \in H(\mathrm{div},\Omega;\mathbb{M}),\\ \langle \mathrm{div}\sigma, v\rangle &= \langle f, v\rangle, \quad v \in L^2(\Omega;\mathbb{R}^2),\\ \langle S_1\sigma, q\rangle &= 0, \quad q \in L^2(\Omega). \end{aligned} \tag{4.1}$$

Here $\langle\cdot,\cdot\rangle$ is the standard L^2 inner product on Ω. This problem is well-posed. See [5] and [14] for the proof.

We consider now a finite element discretization of (4.1). For this, we choose families of finite-dimensional subspaces

$$\Lambda_h^1(\mathbb{M}) \subset H(\mathrm{div},\Omega;\mathbb{M}), \Lambda_h^2(\mathbb{R}^2) \subset L^2(\Omega;\mathbb{R}^2), \Lambda_h^2 \subset L^2(\Omega),$$

indexed by h, and seek a discrete solution $(\sigma_h, u_h, p_h) \in \Lambda_h^1(\mathbb{M}) \times \Lambda_h^2(\mathbb{R}^2) \times \Lambda_h^2$ such that

$$\begin{aligned}\langle A\sigma_h,\tau\rangle + \langle \mathrm{div}\tau, u_h\rangle - \langle S_1\tau, p_h\rangle = 0, &\quad \tau \in \Lambda_h^1(\mathbb{M}),\\ \langle \mathrm{div}\sigma_h, v\rangle = \langle f, v\rangle\,, &\quad v \in \Lambda_h^2(\mathbb{R}^2),\\ \langle S_1\sigma_h, q\rangle = 0, &\quad q \in \Lambda_h^2.\end{aligned} \tag{4.2}$$

The stability of (4.2) will be ensured by the Brezzi stability conditions:

$$\text{(S1)}\ \ \|\tau\|^2_{H(\mathrm{div},\Omega;\mathbb{M})} \le c_1(A\tau,\tau) \text{ whenever } \tau\in\Lambda_h^1(\mathbb{R}^2) \text{ satisfies } \langle\mathrm{div}\tau, v\rangle = 0\ \ \forall v \in \Lambda_h^2(\mathbb{R}^2) \text{ and } \langle S_1\tau, q\rangle = 0\ \forall q\in\Lambda_h^2, \tag{4.3}$$

$$\text{(S2) for all nonzero } (v,q)\in\Lambda_h^2(\mathbb{R}^2)\times\Lambda_h^2, \text{ there exists nonzero } \tau\in\Lambda_h^1(\mathbb{R}^2) \text{ with } \langle\mathrm{div}\tau, v\rangle - \langle S_1\tau, q\rangle \ge c_2\,\|\tau\|_{H(\mathrm{div},\Omega;\mathbb{M})}\,(\|v\|_{L^2(\Omega;\mathbb{R}^2)} + \|q\|_{L^2(\Omega)}), \tag{4.4}$$

where constants c_1 and c_2 are independent of h.

For meshes of arbitrary but uniform order, conditions (4.3) and (4.4) have been proved in [5] and [14]. In what follows, we will demonstrate that they are also satisfied for (2D) meshes with elements of variable (but limited) order.

Before presenting our proof, we would like to comment shortly on difficulties encountered in proving stability for generalizing AFW elements with variable order. As we have shown in Section 3, it is rather straightforward to generalize AFW elements to the variable order case. The following commuting diagrams are essential in the stability proof from [6, 5, 14],

(4.5)

$$\begin{tikzcd} H^1(\Omega;\mathbb{R}^2) \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi_h^{1,-}"'] & L^2(\Omega) \arrow[d, "\Pi_h^2"] \\ \mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T}_h) \arrow[r, "\mathrm{div}"] & \mathcal{P}_r\Lambda^2(\mathcal{T}_h) \end{tikzcd} \quad \begin{tikzcd} H^1(\Omega;\mathbb{R}^2) \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi_h^1"'] & L^2(\Omega) \arrow[d, "\Pi_h^2"] \\ \mathcal{P}_{r+1}\Lambda^1(\mathcal{T}_h) \arrow[r, "\mathrm{div}"] & \mathcal{P}_r\Lambda^2(\mathcal{T}_h) \end{tikzcd} \quad \begin{tikzcd} H^1(\Omega;\mathbb{R}^2) \arrow[r, "Id"] \arrow[d, "\Pi_h^0"'] & H^1(\Omega;\mathbb{R}^2) \arrow[d, "\Pi_h^{1,-}"] \\ \mathcal{P}_{r+2}\Lambda^0(\mathcal{T}_h;\mathbb{R}^2) \arrow[r, "\Pi_h^{1,-}\circ Id"] & \mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T}_h) \end{tikzcd}$$

When uniform order r is replaced by variable order $\tilde{r}$, then the left and middle diagrams do not commute if $\Pi_h^{1,-}$ and Π_h^1 are natural generalizations of canonical projection operators introduced in [6, 5, 14]. A counterexample is given in the appendix of [19]. To our best knowledge, the only operators which make these two diagrams commute for meshes with variable order, are projection based interpolation operators introduced in [19]. But then we need to construct an operator Π_h^0 that makes the right diagram commute if $\Pi_h^{1,-}$ is the projection based interpolation operator. We replaced the operator Π_h^0 with a new operator W_h, constructed in [19], but we were able to prove its well-definedness only for $0 \le \tilde{r}(T) \le 3$. In the following sections we pursue a different strategy, defining *new* projection based interpolation $\Pi_h^{1,-}, \Pi_h^1$ operators, and a new operator W_h in such a way that we can make all three diagrams commute.

5. Preliminaries for the proof of stability

We begin by recalling our assumptions on the domain and meshes: Ω is a polygon and it is meshed with a family $(\mathcal{T}_h)_h$ of affine triangular meshes satisfying assumption of regularity. For any mesh $\mathcal{T}_h$, mapping $\tilde{r} : \triangle(\mathcal{T}_h) \to \mathbb{Z}_+$ defines a locally variable order of discretization that satisfies the minimum rule, $\tilde{r}(e) \leq \tilde{r}(f)$ for $e \subset f,\ e, f \in \triangle_T$. The maximum order is limited, i.e. $\sup_h \sup_{T\in\mathcal{T}_h} \tilde{r}(T) < \infty$.

Definition 5.1. For any $T \in \mathcal{T}_h$, we define a linear operator $\Pi^2_{\tilde{r},T}$ as the standard orthogonal projection operator from $L^2(T)$ onto $\mathcal{P}_{\tilde{r}}\Lambda^2(T)$.

5.1. Projection Based Interpolation onto $\mathcal{P}_{\tilde{r}+1}\Lambda^1(T)$.

Definition 5.2. For any $T \in \mathcal{T}_h$, we define a linear operator $\Pi^1_{\tilde{r}+1,T} : H^1(T;\mathbb{R}^2) \longrightarrow \mathcal{P}_{\tilde{r}+1}\Lambda^1(T)$ by the relations

$$\int_T \operatorname{div}(\Pi^1_{\tilde{r}+1,T}\omega - \omega)(\mathbf{x})\psi(\mathbf{x})d\mathbf{x} = 0 \quad \forall\psi \in \mathcal{P}_{\tilde{r}(T)}(T)/\mathbb{R} \tag{5.1}$$

$$\int_T (\Pi^1_{\tilde{r}+1,T}\omega(\mathbf{x}) - \omega(\mathbf{x}))^\top \operatorname{curl}\varphi(\mathbf{x})d\mathbf{x} = 0 \quad \forall\varphi \in \mathring{\mathcal{P}}_{\tilde{r}(T)+2}(T) \tag{5.2}$$

$$\int_e [(\Pi^1_{\tilde{r}+1,T}\omega - \omega)\cdot\mathbf{n}]\eta(s)ds = 0 \quad \forall\eta \in \mathcal{P}_{\tilde{r}(e)+1}(e)\,, \forall e \in \triangle_1(T) \tag{5.3}$$

Here $\mathbf{n}$ is a unit outward normal vector along e.

The operator $\Pi^1_{\tilde{r}+1,T}$ is the Projection Based Interpolation operator onto $\mathcal{P}_{\tilde{r}+1}\Lambda^1(T)$ defined in [19]. We immediately have the following two lemmas.

Lemma 5.3. *For any $T \in \mathcal{T}_h$, and any $\omega \in [H^1(T)]^2$, we have*

$$\Pi^2_{\tilde{r},T}\, div\omega = div\Pi^1_{\tilde{r}+1,T}\omega.$$

Lemma 5.4. *There exists $C > 0$ such that*

$$\|\Pi^1_{\tilde{r},T}\omega\|_{L^2(T)} \leq C\|\omega\|_{H^1(T)} \quad \forall T \in \mathcal{T}_h, \omega \in H^1(T;\mathbb{R}^2).$$

5.2. Modified Projection Based Interpolation onto $\mathcal{P}^-_{\tilde{r}+1}\Lambda^1(T)$ and modified operator W onto $\mathcal{P}_{\tilde{r}+2}\Lambda^0(T)$.

In order to prove the stability of the mixed FE method, we need to make the left and right diagrams in (4.5) commute. In [19], the definition of Projection Based (PB) interpolation operator onto $\mathcal{P}^-_{\tilde{r}+1}\Lambda^1(\hat{T})$ was very similar to $\mathcal{P}_{\tilde{r}+1}\Lambda^1(\hat{T})$. From the proof of Lemma 10 in [19], we can see that only conditions (5.1) and (5.3) are used to prove the commutativity of the middle diagram in (4.5). This implies that we may be able to change condition (5.2) for the PB interpolation operator onto $\mathcal{P}^-_{\tilde{r}+1}\Lambda^1(\hat{T})$ in such a way that we can design a new operator W_h which makes the right diagram in (4.5) commute, and can be proved to be well-defined, for an arbitrary order of discretization.

Definition 5.5. Let $\tilde{r} : \triangle(\hat{T}) \to \mathbb{Z}_+$ be a mapping that prescribes the local order of discretization and satisfies the minimum rule, i.e. if $\hat{e}, \hat{f} \in \triangle(\hat{T})$ and $\hat{e} \subset \hat{f}$ then $\tilde{r}(\hat{e}) \leq \tilde{r}(\hat{f})$. We put $k_{\tilde{r}} = \dim \operatorname{curl}_{\hat{\mathbf{x}}}\mathring{\mathcal{P}}_{\tilde{r}(\hat{T})+1}(\hat{T})$. Let $\{\hat{\mathbf{f}}_{\tilde{r},1}(\hat{\mathbf{x}}), \cdots, \hat{\mathbf{f}}_{\tilde{r},k_{\tilde{r}}}(\hat{\mathbf{x}})\}$ be a basis of $\operatorname{curl}_{\hat{\mathbf{x}}}\mathring{\mathcal{P}}_{\tilde{r}(\hat{T})+1}(\hat{T})$. Let $\{\hat{\mathbf{g}}_{\tilde{r},1}(\hat{\mathbf{x}}), \cdots, \hat{\mathbf{g}}_{\tilde{r},k_{\tilde{r}}}(\hat{\mathbf{x}})\}$ be a linearly independent subset of $\mathcal{P}_{\tilde{r}(\hat{T})-1}(\hat{T};\mathbb{R}^2)$ such that

$$\operatorname{span}\{\hat{\mathbf{g}}_{\tilde{r},1}(\hat{\mathbf{x}}), \cdots, \hat{\mathbf{g}}_{\tilde{r},k_{\tilde{r}}}(\hat{\mathbf{x}})\} \oplus \operatorname{grad}_{\hat{\mathbf{x}}}\mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T}) = [\mathcal{P}_{\tilde{r}(\hat{T})-1}(\hat{T})]^2$$

For $t \in [0,1]$, we define $\hat{\mathbf{h}}_{\tilde{r},i}(\hat{\mathbf{x}},t) = (1-t)\hat{\mathbf{f}}_{\tilde{r},1}(\hat{\mathbf{x}}) + t\hat{\mathbf{g}}_{\tilde{r},1}(\hat{\mathbf{x}})$, $1 \le i \le k_{\tilde{r}}$.

Remark 5.6. It is easy to check that $k_{\tilde{r}} = \dim \mathcal{P}_{\tilde{r}(\hat{T})-1}(\hat{T};\mathbb{R}^2) - \dim \mathrm{grad}_{\hat{\mathbf{x}}}\mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T})$.

Remark 5.7. For any $T \in \mathcal{T}_h$, there exists an affine mapping from the reference triangle $\hat{T}$ onto T, which can be written as $\mathbf{x} = B_T\hat{\mathbf{x}} + \mathbf{b}$. Here B_T is a constant nonsingular matrix in $\mathbb{R}^{2\times 2}$. We denote by $\tilde{r}$ a mapping from $\triangle(T)$ to $\mathbb{Z}_+$ such that if $e, f \in \triangle(T)$ and $e \subset f$ then $\tilde{r}(e) \le \tilde{r}(f)$. With the same symbol $\tilde{r}$ we denote the corresponding mapping from $\triangle(\hat{T})$ to $\mathbb{Z}_+$, $\tilde{r}(\hat{f}) := \tilde{r}(f)$ for any $\hat{f} \in \triangle(\hat{T})$, where f is the image of $\hat{f}$ under the affine mapping mentioned above.

Definition 5.8. (One-parameter family of PB interpolation operators onto $\mathcal{P}^-_{\tilde{r}+1}\Lambda^1(\hat{T})$) For any $t \in [0,1]$, we define a linear operator $\Pi^{1,-}_{\tilde{r}+1,\hat{T},t} : H^1(\hat{T};\mathbb{R}^2) \longrightarrow \mathcal{P}^-_{\tilde{r}+1}\Lambda^1(\hat{T})$ by the relations

$$\int_{\hat{T}} \mathrm{div}_{\hat{\mathbf{x}}}(\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega} - \hat{\omega})(\hat{\mathbf{x}})\hat{\psi}(\hat{\mathbf{x}})d\hat{\mathbf{x}} = 0 \quad \forall \hat{\psi} \in \mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T})/\mathbb{R} \tag{5.4}$$

$$\int_{\hat{T}} (\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega}(\hat{\mathbf{x}}) - \hat{\omega}(\hat{\mathbf{x}}))^\top \hat{\mathbf{h}}_i(\hat{\mathbf{x}},t)d\hat{\mathbf{x}} = 0 \quad 1 \le i \le k_{\tilde{r}} \tag{5.5}$$

$$\int_{\hat{e}} [(\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega} - \hat{\omega}) \cdot \hat{\mathbf{n}}]\hat{\eta}d\hat{s} = 0 \quad \forall \hat{\eta} \in \mathcal{P}_{\tilde{r}(\hat{e})}(\hat{e}), \forall \hat{e} \in \triangle_1(\hat{T}) \tag{5.6}$$

Definition 5.9. (One-parameter family of PB interpolation operators onto $\mathcal{P}^-_{\tilde{r}+1}\Lambda^1(T)$) For $t \in [0,1]$, and for any $T \in \mathcal{T}_h$, we define a linear operator $\Pi^{1,-}_{\tilde{r}+1,T,t} : H^1(T;\mathbb{R}^2) \longrightarrow \mathcal{P}^-_{\tilde{r}+1}\Lambda^1(T)$ by the relations

$$\int_T \mathrm{div}(\Pi^{1,-}_{\tilde{r}+1,T,t}\omega - \omega)(\mathbf{x})\hat{\psi}(\hat{\mathbf{x}}(\mathbf{x}))\, d\mathbf{x} = 0 \quad \forall \hat{\psi} \in \mathcal{P}_{\tilde{r}(T)}(\hat{T})/\mathbb{R} \tag{5.7}$$

$$\int_T (\Pi^{1,-}_{\tilde{r}+1,T,t}\omega(\mathbf{x}) - \omega(\mathbf{x}))^\top B_T^{-\top}\hat{\mathbf{h}}_i(\hat{\mathbf{x}}(\mathbf{x}),t)d\mathbf{x} = 0 \quad 1 \le i \le k_{\tilde{r}} \tag{5.8}$$

$$\int_e [(\Pi^{1,-}_{\tilde{r}+1,T,t}\omega - \omega) \cdot \mathbf{n}]\eta(s)ds = 0 \quad \forall \eta \in \mathcal{P}_{\tilde{r}(e)}(e), \forall e \in \triangle_1(T) \tag{5.9}$$

In the above, $\hat{\mathbf{x}} = \hat{\mathbf{x}}(\mathbf{x})$ denotes the inverse of the affine mapping introduced in Remark 5.7. The matrix B_T is defined in Remark 5.7 as well. And according to Remark 5.7, $\tilde{r}(f) = \tilde{r}(\hat{f})$ for any $\hat{f} \in \triangle(\hat{T})$, where f is the image of $\hat{f}$ under the affine mapping from $\hat{T}$ to T.

Definition 5.10. In the sequel, the phrase "for almost all" (parameters) will mean "for all except for a finite number" (of parameters). For example, a sequence x_n in a topological space converges to x if, for every neighborhood of x, almost all values of x_n belong to the neighborhood.

Lemma 5.11. $\Pi^{1,-}_{\tilde{r}+1,\hat{T},t} : H^1(\hat{T};\mathbb{R}^2) \longrightarrow \mathcal{P}^-_{\tilde{r}+1}\Lambda^1(\hat{T})$ *is a well-defined linear operator for almost all* $t \in [0,1]$*. Moreover,*

$$div_{\hat{\mathbf{x}}}\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega} = \Pi^2_{\tilde{r},\hat{T}}div_{\hat{\mathbf{x}}}\hat{\omega}$$

for any $\hat{\omega} \in H^1(\hat{T};\mathbb{R}^2)$.

Proof. The linearity of the operator is obvious, for any $t \in \mathbb{R}$. The point is to show that the operator is well-defined. For $t = 0$, the operator $\Pi^{1,-}_{\tilde{r}+1,\hat{T},0}$ reduces to the PB interpolation defined in [19] and, according to Lemma 9 and Lemma 20 from [19], is well-defined. Moreover,

- $\int_{\hat{T}} \mathrm{div}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})\hat{\psi}(\hat{\mathbf{x}})d\hat{x}_1 d\hat{x}_2$ is a continuous functional of $\hat{\omega}$, for any $\hat{\psi} \in \mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T})/\mathbb{R}$,
- $\int_{\hat{T}} \hat{\omega}(\hat{\mathbf{x}})^\top \hat{\mathbf{h}}_i(\hat{\mathbf{x}}, t)d\hat{x}_1 d\hat{x}_2$ is a continuous functional of $\hat{\omega}$, for any $1 \le i \le k_{\tilde{r}}$ and any $t \in \mathbb{R}$, and
- $\int_{\hat{e}} [\hat{\omega} \cdot \hat{\mathbf{n}}]\hat{\eta} d\hat{s}$ is a continuous functional of $\hat{\omega}$, for any $\hat{\eta} \in \mathcal{P}_{\tilde{r}(\hat{e})}(\hat{e})$, and any $\hat{e} \in \triangle_1(\hat{T})$.

Therefore, in order to demonstrate that $\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}$ is well-defined, it is sufficient to show that $\hat{\omega} = 0$ if $\hat{\omega} \in \mathcal{P}^-_{\tilde{r}+1}\Lambda^1(\hat{T})$ and $\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega} = 0$.

We take an arbitrary $\hat{\omega} \in \mathcal{P}^-_{\tilde{r}+1}\Lambda^1(\hat{T})$ such that $\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega} = 0$. According to the definition of $\mathcal{P}^-_{\tilde{r}+1}\Lambda^1(\hat{T})$ and (5.6), we know that $\hat{\omega} \in \mathring{\mathcal{P}}^-_{\tilde{r}+1}\Lambda^1(\hat{T})$. Set $r = \tilde{r}(\hat{T})$. We denote by $C(t, r)$ the matrix associated with conditions (5.4) and (5.5), corresponding to a particular basis for $\mathring{\mathcal{P}}^-_{\tilde{r}+1}\Lambda^1(\hat{T})$ (the solution space), and a particular basis for $\mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T})/\mathbb{R}$. We argue that matrix $C(t, r)$ is non-singular for almost all $t \in [0, 1]$. Notice that, for any r, $\det(C(t, r))$ is a polynomial in t. Since $\Pi^{1,-}_{\tilde{r}+1,\hat{T},0}$ is well-defined, $\det(C(0, r)) \neq 0$. So $\det(C(t, r))$ is a non-zero polynomial. By the fundamental theorem of algebra, the polynomial has a finite number of real roots. We can conclude thus that $\Pi^{1,-}_{\tilde{r}+1,\hat{T},t} : H^1(\hat{T}; \mathbb{R}^2) \longrightarrow \mathcal{P}^-_{\tilde{r}+1}\Lambda^1(\hat{T})$ is well-defined for any $t \in [0, 1]$ except for the roots of $\det(C(t, r))$. The number of roots is independent of the choice of the bases and depends only upon $r = \tilde{r}(\hat{T})$.

Since $\mathrm{div}_{\hat{\mathbf{x}}}\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega} \in \mathcal{P}_{\tilde{r}}\Lambda^2(\hat{T})$, for any $\hat{\omega} \in H^1(\hat{T}; \mathbb{R}^2)$, then $\Pi^2_{\tilde{r},\hat{T}}\mathrm{div}_{\hat{\mathbf{x}}}\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega} = \mathrm{div}_{\hat{\mathbf{x}}}\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega}$. In order to show that $\mathrm{div}_{\hat{\mathbf{x}}}\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega} = \Pi^2_{\tilde{r},\hat{T}}\mathrm{div}_{\hat{\mathbf{x}}}\hat{\omega}$, it is sufficient to show that $\Pi^2_{\tilde{r},\hat{T}}\mathrm{div}_{\hat{\mathbf{x}}}\hat{\omega} = 0$, for any $\hat{\omega}$ such that $\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega} = 0$. We have

$$\int_{\hat{T}} \mathrm{div}_{\hat{\mathbf{x}}}\hat{\omega}\hat{\psi}d\hat{\mathbf{x}} = \int_{\hat{T}} \mathrm{div}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\chi} + c)d\hat{\mathbf{x}} = \int_{\hat{T}} \mathrm{div}_{\hat{\mathbf{x}}}\hat{\omega}\hat{\chi}d\hat{\mathbf{x}} + c\int_{\partial\hat{T}} \hat{\omega} \cdot \hat{\mathbf{n}}d\hat{s} = 0.$$

Here $\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega} = 0$, $\hat{\psi} \in \mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T})$, $c = \int_{\hat{T}} \hat{\psi}d\hat{\mathbf{x}}$. The last equality holds due to the definition of $\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}$ and the fact that $\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega} = 0$. We have thus $\mathrm{div}_{\hat{\mathbf{x}}}\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega} = \Pi^2_{\tilde{r},\hat{T}}\mathrm{div}_{\hat{\mathbf{x}}}\hat{\omega}$ for any $\hat{\omega} \in H^1(\hat{T}; \mathbb{R}^2)$. □

Theorem 5.12. *Let $t \in [0, 1]$ be any value for which $\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}$ is well-defined. Then the operator $\Pi^{1,-}_{\tilde{r}+1,T,t} : H^1(T; \mathbb{R}^2) \longrightarrow \mathcal{P}^-_{\tilde{r}+1}\Lambda^1(T)$ is well-defined as well, and we have the following result.*

There exist $C > 0$ such that,for $T \in \mathcal{T}_h$, and $\omega \in H^1(T; \mathbb{R}^2)$, the corresponding function $\hat{\omega}$ defined by

$$\omega(\mathbf{x}(\hat{\mathbf{x}})) = \frac{B_T}{\det(B_T)}\hat{\omega}(\hat{\mathbf{x}}) \tag{5.10}$$

belongs to $H^1(\hat{T}; \mathbb{R}^2)$, and

$$\Pi^{1,-}_{\tilde{r}+1,T,t}\omega(\mathbf{x}(\hat{\mathbf{x}})) = \frac{B_T}{\det(B_T)}\Pi^{1,-}_{\tilde{r}+1,\hat{T},t}\hat{\omega}(\hat{\mathbf{x}}) \tag{5.11}$$

Additionally,

$$div\Pi^{1,-}_{\tilde{r}+1,T,t}\omega = \Pi^2_{\tilde{r},T}div\omega \tag{5.12}$$

and

$$\|\Pi^{1,-}_{\tilde{r}+1,T,t}\omega\|_{L^2(T)} \leq C\|\omega\|_{H^1(T)} \tag{5.13}$$

Proof. It is easy to see that $\hat{\omega} \in H^1(\hat{T};\mathbb{R}^2)$. We define a linear isomorphism A_T from $H^1(\hat{T};\mathbb{R}^2)$ to $H^1(T;\mathbb{R}^2)$ by $(A_T\hat{\omega})(\mathbf{x}(\hat{\mathbf{x}})) = \dfrac{B_T}{\det(B_T)}\hat{\omega}(\hat{\mathbf{x}})$. It is easy to see that A_T is a linear isomorphism from $\mathcal{P}^-_{\tilde{r}+1}\Lambda^1(\hat{T})$ to $\mathcal{P}^-_{\tilde{r}+1}\Lambda^1(T)$.

By pulling back to $\hat{T}$ by A_T (applied to both ω and $\Pi^{1,-}_{\tilde{r}+1,T,t}\omega$), we can see that (5.7) is the same as (5.4), (5.8) is the same as (5.5), and (5.9) is the same as (5.6). So we have (5.11). As a consequence of (5.11), (5.10) and Lemma 5.11, we have (5.12). Inequality (5.13) is proved by using standard scaling techniques. □

Definition 5.13. We define a linear operator $C_t : [H^1(\hat{T})]^2 \longrightarrow \mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$ by the following relations

$$\int_{\hat{T}} \mathrm{div}_{\hat{\mathbf{x}}} C_t\hat{\omega}(\hat{\mathbf{x}})\hat{\psi}(\hat{\mathbf{x}})d\hat{\mathbf{x}} = \int_{\hat{T}} \mathrm{div}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})\hat{\psi}(\hat{\mathbf{x}})d\hat{\mathbf{x}} \quad \forall\hat{\psi} \in \mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T})/\mathbb{R} \tag{5.14}$$

$$\int_{\hat{T}} (C_t\hat{\omega}(\hat{\mathbf{x}}))^\top \hat{\mathbf{h}}_i(\hat{\mathbf{x}},t)d\hat{\mathbf{x}} = \int_{\hat{T}} (\hat{\omega}(\hat{\mathbf{x}}))^\top \hat{\mathbf{h}}_i(\hat{\mathbf{x}},t)d\hat{\mathbf{x}} \quad 1 \leq i \leq k_{\tilde{r}} \tag{5.15}$$

$$\int_{\hat{e}} [(C_t\hat{\omega})\cdot\hat{\mathbf{n}}]\hat{\eta}d\hat{s} = \int_{\hat{e}} [\hat{\omega}\cdot\hat{\mathbf{n}}]\hat{\eta}d\hat{s} \quad \forall\hat{\eta} \in \mathcal{P}_{\tilde{r}(\hat{e})}(\hat{e}), \forall\hat{e} \in \triangle_1(\hat{T}) \tag{5.16}$$

$$\int_{\hat{e}} [(C_t\hat{\omega})\cdot\hat{\mathbf{t}}]\hat{\eta}d\hat{s} = \int_{\hat{e}} [\hat{\omega}\cdot\hat{\mathbf{t}}]\hat{\eta}d\hat{s} \quad \forall\hat{\eta} \in \mathcal{P}_{\tilde{r}(\hat{e})}(\hat{e}), \forall\hat{e} \in \triangle_1(\hat{T}) \tag{5.17}$$

$$C_t\hat{\omega} = 0 \quad \text{at all vertices of } \hat{T} \tag{5.18}$$

Here $\hat{\mathbf{n}}, \hat{\mathbf{t}}$ denote the normal and tangent unit vectors along $\partial\hat{T}$.

Definition 5.14. For any $T \in \mathcal{T}_h$, we define a linear operator $W_{T,t} : H^1(T;\mathbb{R}^2) \longrightarrow \mathcal{P}_{\tilde{r}+2}\Lambda^0(T;\mathbb{R}^2)$ by the following relations

$$\int_T \mathrm{div}(W_{T,t}\omega - \omega)(\mathbf{x})\hat{\psi}(\hat{\mathbf{x}}(\mathbf{x}))d\mathbf{x} = 0 \quad \forall\hat{\psi} \in \mathcal{P}_{\tilde{r}(T)}(\hat{T})/\mathbb{R} \tag{5.19}$$

$$\int_T (W_{T,t}\omega(\mathbf{x}) - \omega(\mathbf{x}))^\top B_T^{-\top}\hat{\mathbf{h}}_i(\hat{\mathbf{x}}(\mathbf{x}),t)d\mathbf{x} = 0 \quad 1 \leq i \leq k_{\tilde{r}} \tag{5.20}$$

$$\int_e [(W_{T,t}\omega - \omega)\cdot\mathbf{n}]\eta ds = 0 \quad \forall\eta \in \mathcal{P}_{\tilde{r}(e)}(e),\ \forall e \in \triangle_1(T) \tag{5.21}$$

$$\int_e [(W_{T,t}\omega - \omega)\cdot\mathbf{t}]\eta ds = 0 \quad \forall\eta \in \mathcal{P}_{\tilde{r}(e)}(e),\ \forall e \in \triangle_1(T) \tag{5.22}$$

$$W_{T,t}\omega = 0 \text{ at all vertices of } T \tag{5.23}$$

Here $\mathbf{n}, \mathbf{t}$ denote the normal and tangent unit vectors along ∂T. As before, $\hat{\mathbf{x}} = \hat{\mathbf{x}}(\mathbf{x})$ stands for the inverse of the affine mapping from the master to the physical element, defined along with matrix B_T in Remark 5.7. Also, according to Remark 5.7,

$\tilde{r}(f) = \tilde{r}(\hat{f})$ for any $\hat{f} \in \triangle(\hat{T})$, where f is the image of $\hat{f}$ under the affine mapping from $\hat{T}$ to T.

Lemma 5.15. *Operator $C_t : H^1(\hat{T};\mathbb{R}^2) \longrightarrow \mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$ is well-defined for almost all $t \in [0,1]$.*

Proof. Take an arbitrary $t \in [0,1]$. Since $\hat{\omega} \in H^1(\hat{T};\mathbb{R}^2)$, then all right hand sides of (5.14),(5.15),(5.16), (5.17) are continuous functionals with respect to $\hat{\omega}$. In order to show the well-definedness of C_t, it is sufficient to show that $\hat{\omega} = 0$ if $\hat{\omega} \in \mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$, $\hat{\omega} = 0$ at all vertices of $\hat{T}$, and $C_t\hat{\omega} = 0$. According to conditions (5.16) and (5.17), $\hat{\omega} \in \mathring{\mathcal{P}}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$. Define now $\mathring{C}_t : H^1(\hat{T};\mathbb{R}^2) \longrightarrow \mathring{\mathcal{P}}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$ by the relations

$$\int_{\hat{T}} \mathrm{div}_{\hat{\mathbf{x}}}\mathring{C}_t\hat{\omega}(\hat{\mathbf{x}})\hat{\psi}(\hat{\mathbf{x}})d\hat{\mathbf{x}} = \int_{\hat{T}} \mathrm{div}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})\hat{\psi}(\hat{\mathbf{x}})d\hat{\mathbf{x}} \quad \forall\hat{\psi}(\hat{\mathbf{x}}) \in \mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T})/\mathbb{R} \tag{5.24}$$

$$\int_{\hat{T}} (\mathring{C}_t\hat{\omega}(\hat{\mathbf{x}}))^\top \hat{\mathbf{h}}_i(\hat{\mathbf{x}},t)d\hat{\mathbf{x}} = \int_{\hat{T}} (\hat{\omega}(\hat{\mathbf{x}}))^\top \hat{\mathbf{h}}_i(\hat{\mathbf{x}},t)d\hat{\mathbf{x}} \quad 1 \le i \le k_{\tilde{r}} \tag{5.25}$$

It is sufficient to show that operator $\mathring{C}_{T,t} : H^1(\hat{T};\mathbb{R}^2) \longrightarrow \mathring{\mathcal{P}}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$ is well-defined.

Obviously, the right-hand side of conditions (5.24) and (5.25) are continuous functionals with respect to $\hat{\omega}$. Set $r = \tilde{r}(\hat{T})$ and denote by $\mathring{C}(t,r)$ the matrix associated with the left-hand side of conditions (5.24) and (5.25) corresponding to some basis of $\mathring{\mathcal{P}}^-_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$ (the solution space), and some basis of $\mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T})/\mathbb{R}$. We need to show that the $\det(\mathring{C}(t,r)) \neq 0$ for almost all $t \in [0,1]$.

Since $\hat{\omega}$ vanishes on the boundary, we can integrate by parts (5.24) without getting any boundary terms. So (5.24) is the same as

$$\int_{\hat{T}} (\mathring{C}_t(\hat{\omega})(\hat{\mathbf{x}}))^\top \cdot \mathrm{grad}_{\hat{\mathbf{x}}}\hat{\psi}(\hat{\mathbf{x}})d\hat{\mathbf{x}} = \int_{\hat{T}} (\hat{\omega}(\hat{\mathbf{x}}))^\top \cdot \mathrm{grad}_{\hat{\mathbf{x}}}\hat{\psi}(\hat{\mathbf{x}})d\hat{\mathbf{x}} \quad \forall\hat{\psi} \in \mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T})/\mathbb{R}. \tag{5.26}$$

Notice that $\mathring{\mathcal{P}}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2) = \hat{\chi}(\hat{\mathbf{x}})[\mathcal{P}_{\tilde{r}(\hat{T})-1}(\hat{T})]^2$ where $\hat{\chi}$ is a third order polynomial vanishing along boundary $\partial\hat{T}$ and positive in the interior of $\hat{T}$ (a "bubble"). And notice that

$$\begin{aligned}&\{\mathbf{h}_{\tilde{r},1}(\hat{\mathbf{x}},1),\cdots,\mathbf{h}_{\tilde{r},k_{\tilde{r}}}(\hat{\mathbf{x}},1)\} \oplus \mathrm{grad}_{\hat{\mathbf{x}}}\mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T})/\mathbb{R}\\ =&\{\mathbf{g}_{\tilde{r},1}(\hat{\mathbf{x}}),\cdots,\mathbf{g}_{\tilde{r},k_{\tilde{r}}}(\hat{\mathbf{x}})\} \oplus \mathrm{grad}_{\hat{\mathbf{x}}}\mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T}) = [\mathcal{P}_{\tilde{r}(\hat{T})-1}(\hat{T})]^2.\end{aligned}$$

Consequently, $\det(\mathring{C}(1,r)) \neq 0$. Since $\det(\mathring{C}(t,r))$ is a polynomial in t, by the fundamental theorem of algebra argument again, $\det(\mathring{C}(1,r))$ vanishes at a finite number of roots only.

We can conclude thus that operator $C_t : H^1(\hat{T};\mathbb{R}^2) \longrightarrow \mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$ is well-defined for almost all $t \in [0,1]$. Notice that the number of roots is independent of the choice of the bases, and depends upon $\tilde{r}(\hat{T})$ only. $\square$

Theorem 5.16. *For any $t \in [0,1]$ such that $C_t : H^1(\hat{T};\mathbb{R}^2) \longrightarrow \mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$ is well-defined, there exists $C > 0$ such that, for any $T \in \mathcal{T}_h$, operator $W_{T,t} : H^1(T;\mathbb{R}^2) \longrightarrow \mathcal{P}_{\tilde{r}+2}\Lambda^0(T;\mathbb{R}^2)$ is well-defined as well and,*

$$\|\mathit{curl} W_{T,t}\omega\|_{L^2(T)} \le C(h_T^{-1}\|\omega\|_{L^2(T)} + \|\omega\|_{H^1(T)}) \quad \forall\omega \in H^1(T;\mathbb{R}^2). \tag{5.27}$$

Here h_T is the diameter of T.

Proof. For any $T \in \mathcal{T}_h$, we define a linear isomorphism A_T from $H^1(\hat{T};\mathbb{R}^2)$ to $H^1(T;\mathbb{R}^2)$ by $(A_T\hat{\omega})(\mathbf{x}(\hat{\mathbf{x}})) = \dfrac{B_T\hat{\omega}(\hat{\mathbf{x}})}{\det(B_T)}$. It is easy to see that A_T is a linear isomorphism from $\mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$ to $\mathcal{P}_{\tilde{r}+2}\Lambda^0(T;\mathbb{R}^2)$.

By pulling back to $\hat{T}$ by A_T (applied to both ω and $W_{T,t}\omega$), we can see that (5.19) is the same as (5.14), (5.20) is the same as (5.15), (5.21) is the same as (5.16) and (5.23) is the same as (5.18). (5.22) becomes

(5.28)
$$\int_{\hat{e}}[(A_T^{-1}(W_{T,t}\omega))^\top B_T^\top B_T\hat{\mathbf{t}}]\hat{\eta}d\hat{s} = \int_{\hat{e}}[\hat{\omega}^T B_T^\top B_T\hat{\mathbf{t}}]\hat{\eta}d\hat{s} \quad \forall\hat{\eta}\in\mathcal{P}_{\tilde{r}(\hat{e})}(\hat{e}), \forall\hat{e}\in\triangle_1(\hat{T}).$$

Notice that $\hat{\mathbf{n}}$ and $B_T^\top B_T\hat{\mathbf{t}}$ are not parallel to each other. Since $\hat{\mathbf{t}}^\top B_T^\top B_T\hat{\mathbf{t}} \neq 0$, $\hat{\mathbf{t}}$ is not perpendicular to $B_T^\top B_T\hat{\mathbf{t}}$. Since $\hat{\mathbf{n}}\perp\hat{\mathbf{t}}$, then $\hat{\mathbf{n}}$ is not parallel to $B_T^\top B_T\hat{\mathbf{t}}$. This means that conditions (5.21) and (5.22) imply conditions (5.16) and (5.17) by the definition of A_T. So we can conclude that $W_{T,t}$ is well-defined when C_t is well-defined, and

$$W_{T,t}\omega(\mathbf{x}(\hat{\mathbf{x}})) = A_TC_t\hat{\omega}(\hat{\mathbf{x}}). \tag{5.29}$$

Finally, inequality (5.27) results from standard scaling techniques. □

Definition 5.17. Let $\tilde{r} : \triangle(\hat{T}) \to \mathbb{Z}_+$ be a locally variable order of discretization that satisfies the minimum rule. According to Lemma 5.11 and Lemma 5.15, there exists $t_{\tilde{r}(\hat{T})} \in [0,1]$ such that both $\Pi^{1,-}_{\tilde{r}+1,\hat{T},t_{\tilde{r}(\hat{T})}}$ and $C_{t_{\tilde{r}(\hat{T})}}$ are well-defined. And the value of $t_{\tilde{r}(\hat{T})}$ depends on $\tilde{r}(\hat{T})$ only.

5.3. **Projection operators on the whole mesh.** According to Lemma 5.11, Theorem 5.12, Lemma 5.15, and Theorem 5.16, there exist $\{t_1, t_2, \cdots\} \subset \mathbb{Z}_+$ such that, for any $T \in \mathcal{T}_h$, $\Pi^{1,-}_{\tilde{r}+1,T,t_{\tilde{r}(T)}}$ and $W_{T,t_{\tilde{r}(T)}}$ are well-defined linear operators. From now on, we rename $\Pi^{1,-}_{\tilde{r}+1,T,t_{\tilde{r}(T)}}$ by $\Pi^{1,-}_{\tilde{r}+1,T}$, and $W_{T,t_{\tilde{r}(T)}}$ by W_T.

Definition 5.18. We define the following global interpolation operators,

$$\Pi^2_{\tilde{r},\mathcal{T}_h} : L^2(\Omega) \longrightarrow \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h), \quad (\Pi^2_{\tilde{r},\mathcal{T}_h}u)|_T = \Pi^2_{\tilde{r},T}(u|_T)$$

$$\tilde{\Pi}^2_{\tilde{r},\mathcal{T}_h} : L^2(\Omega;\mathbb{R}^2) \longrightarrow \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h;\mathbb{R}^2), \quad (\tilde{\Pi}^2_{\tilde{r},\mathcal{T}_h}(u_1,u_2)^\top)|_T = (\Pi^2_{\tilde{r},T}(u_1|_T), \Pi^2_{\tilde{r},T}(u_2|_T))^\top$$

$$\Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h} : H^1(\Omega;\mathbb{R}^2) \longrightarrow \mathcal{P}^{-1}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h), \quad (\Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}\omega)|_T = \Pi^{1,-}_{\tilde{r}+1,T}(\omega|_T)$$

$$\tilde{\Pi}^1_{\tilde{r}+1,\mathcal{T}_h} : H^1(\Omega;\mathbb{M}) \longrightarrow \mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h;\mathbb{R}^2), \quad (\tilde{\Pi}^1_{\tilde{r}+1,\mathcal{T}_h}\sigma)|_T = \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix}$$

where

$$\begin{bmatrix} \tau_{11} \\ \tau_{12} \end{bmatrix} = \Pi^1_{\tilde{r}+1,\mathcal{T}_h}\begin{bmatrix} \sigma_{11}|_T \\ \sigma_{12}|_T \end{bmatrix}, \quad \begin{bmatrix} \tau_{21} \\ \tau_{22} \end{bmatrix} = \Pi^1_{\tilde{r}+1,\mathcal{T}_h}\begin{bmatrix} \sigma_{21}|_T \\ \sigma_{22}|_T \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$$W_{\mathcal{T}_h} : H^1(\Omega;\mathbb{R}^2) \longrightarrow \mathcal{P}_{\tilde{r}+2}\Lambda^0(\mathcal{T}_h;\mathbb{R}^2), \quad (W_{\mathcal{T}_h}\omega)|_T = W_T(\omega|_T)$$

for all $T \in \mathcal{T}_h$.

Theorem 5.19. *There exists $C > 0$ such that*

$$\|\tilde{\Pi}^1_{\tilde{r}+1,\mathcal{T}_h}\sigma\|_{L^2(\Omega)} \leq C\|\sigma\|_{H^1(\Omega)}, \quad \|\Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}\omega\|_{L^2(\Omega)} \leq C\|\omega\|_{H^1(\Omega)},$$

for any $\sigma \in H^1(\Omega;\mathbb{M})$ and $\omega \in H^1(\Omega;\mathbb{R}^2)$. Moreover,

$$div\tilde{\Pi}^1_{\tilde{r}+1,\mathcal{T}_h}\sigma = \tilde{\Pi}^2_{\tilde{r},\mathcal{T}_h}div\sigma, \quad div\Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}\omega = \Pi^2_{\tilde{r},\mathcal{T}_h}div\omega$$

Proof. This is an immediate result of Lemma 5.4 and Theorem 5.12. □

Definition 5.20. Let R_h denote the generalized Clement interpolant operator from Theorem 5.1 in [7], mapping $H^1(\Omega;\mathbb{R}^2)$ into $\mathcal{P}_1\Lambda^0(\mathcal{T}_h;\mathbb{R}^2)$. We define

$$\tilde{W}_h = W_h(I - R_h) + R_h$$

Theorem 5.21. *There exists $C > 0$ such that*

$$\|curl\tilde{W}_h\omega\|_{L^2(\Omega)} \leq C\|\omega\|_{H^1(\Omega)} \quad \forall \omega \in H^1(\Omega;\mathbb{R}^2)$$

Operator $\tilde{W}_h$ maps $H^1(\Omega;\mathbb{R}^2)$ into $\mathcal{P}_{\tilde{r}+2}\Lambda^0(T;\mathbb{R}^2)$ and satisfies the condition

$$\Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}\omega = \Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}\tilde{W}_h\omega \quad \forall \omega \in H^1(\Omega;\mathbb{R}^2)$$

Proof. We utilize Example 1 from [7] (in this case, R_h is the same as standard Clement operator) with uniform order equal 1 to construct operator R_h. Operator R_h maps $H^1(\Omega;\mathbb{R}^2)$ into $\mathcal{P}_1\Lambda^0(T;\mathbb{R}^2) \subset \mathcal{P}_{\tilde{r}+2}\Lambda^0(T;\mathbb{R}^2)$. According to Theorem 5.1 from [7], there exists a constant $c > 0$ such that, for any $T \in \mathcal{T}_h$,

$$\|\omega - R_h\omega\|_{L^2(T)} + h_T\|\omega - R_h\omega\|_{H^1(T)} \leq ch_T\|\omega\|_{H^1(K_T)} \quad \forall \omega \in H^1(\Omega;\mathbb{R}^2) \tag{5.30}$$

where K_T stands for the patch of elements adjacent to T, $K_T = \bigcup_{T'\in\mathcal{T}_h, T'\cap T\neq\emptyset} T'$.

As $(\mathcal{T}_h)_h$ is regular, $\sup_h \sup_{T\in\mathcal{T}_h} \#\{T' \in \mathcal{T}_h : T' \subset K_T\} < \infty$. We have

$$\begin{aligned}
\|\mathrm{curl}\tilde{W}_h\omega\|_{L^2(\Omega)} \leq & \|\mathrm{curl}W_h(I - R_h)\omega\|_{L^2(\Omega)} + \|\mathrm{curl}R_h\omega\|_{L^2(\Omega)} \\
\leq & c(h_T^{-1}\|(I - R_h)\omega\|_{L^2(\Omega)} + \|(I - R_h)\omega\|_{H^1(\Omega)} + \|\mathrm{curl}R_h\omega\|_{L^2(\Omega)}) \\
\leq & C\|\omega\|_{H^1(\Omega)} \quad \forall \omega \in H^1(\Omega;\mathbb{R}^2).
\end{aligned}$$

The second inequality above holds by Theorem 5.16 and the third one by (5.30).

According to the definition of $\Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}$, and the definition of W_h, $\Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}\omega = \Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}W_h\omega$, for any $\omega \in H^1(\Omega;\mathbb{R}^2)$. Notice that $(I - R_h)\omega \in \mathcal{P}_1\Lambda^0(\mathcal{T}_h) \subset H^1(\Omega)$. This implies

$$\Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}(I - \tilde{W}_h)\omega = \Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}(I - W_h)(I - R_h)\omega = 0$$

So $\Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}\omega = \Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}\tilde{W}_h\omega$. □

6. Stability of the finite element discretization

We need the following well-known result from partial differential equations, see e.g. [16].

Lemma 6.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary. Then, for all $\mu \in L^2(\Omega)$, there exists $\eta \in H^1(\Omega;\mathbb{R}^2)$ satisfying $div\eta = \mu$. If, in addition, $\int_\Omega \mu = 0$, then we can choose $\eta \in \mathring{H}^1(\Omega)$.*

The main result of this paper for affine meshes is Theorem 6.3 below. Its proof follows the lines of Theorem 9.1 from [14], Theorem 7.1 from [6] and Theorem 11.4 from [5]. The main difference is the use of operator $\tilde{W}_h$ in place of the operator $\tilde{\Pi}^{n-2}_h$ in [14].

Lemma 6.2. *There exists $c > 0$ such that, for any $(\omega, \mu) \in \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h)\times\mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h;\mathbb{R}^2)$, there exists $\sigma \in \mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h;\mathbb{R}^2)$ such that*

$$div\sigma = \mu, \quad -\Pi^2_{\tilde{r},\mathcal{T}_h} S_1\sigma = \omega$$

and

$$\|\sigma\|_{H(div,\Omega)} \leq c(\|\mu\|_{L^2(\Omega)} + \|\omega\|_{L^2(\Omega)})$$

Here, the constant c depends on $\sup_h \sup_{T\in\mathcal{T}_h} \tilde{r}(T)$.

Proof. (1) By Lemma 6.1, we can find $\eta \in H^1(\Omega;\mathbb{M})$ with $\text{div}\eta = \mu$ and $\|\eta\|_{H^1(\Omega)} \leq c\|\mu\|_{L^2(\Omega)}$.

(2) Since $\omega + \Pi^2_{\tilde{r},h} S_1\tilde{\Pi}^1_{\tilde{r}+1,h}\eta \in L^2(\Omega)$, we can apply Lemma 6.1 again to find $\tau \in H^1(\Omega;\mathbb{R}^2)$ with $\text{div}\tau = \omega + \Pi^2_{\tilde{r},h} S_1\tilde{\Pi}^1_{\tilde{r}+1,h}\eta$ such that

$$\|\tau\|_{H^1(\Omega)} \leq c(\|\omega\|_{L^2(\Omega)} + \|\Pi^2_{\tilde{r},h} S_1\tilde{\Pi}^1_{\tilde{r}+1,h}\eta\|_{L^2(\Omega)})$$

(3) Define $\sigma = \text{curl}\tilde{W}_h\tau + \tilde{\Pi}^1_{\tilde{r}+1,h}\eta \in \mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h;\mathbb{R}^2)$. $\sigma \in \mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h;\mathbb{R}^2)$ by Lemma 3.5.

(4) From Step (3), Theorem 5.19, Step (1), and the fact that $\tilde{\Pi}^2_{\tilde{r},h}\tilde{\Pi}^2_{\tilde{r},h} = \tilde{\Pi}^2_{\tilde{r},h}$, we obtain

$$\text{div}\sigma = \text{div}\tilde{\Pi}^1_{\tilde{r}+1,h}\eta = \tilde{\Pi}^2_{\tilde{r},h}\text{div}\eta = \tilde{\Pi}^2_{\tilde{r},h}\mu = \mu.$$

(5) Notice that $\text{div}\omega = -S_1\text{curl}\omega$, for any $\omega \in H^1(\Omega;\mathbb{R}^2)$. Here operator S_1 is defined in (1.10). Then $\Pi^2_{\tilde{r},h}\text{div}\Pi^{1,-}_{\tilde{r}+1,h}\omega = -\Pi^2_{\tilde{r},h}S_1\text{curl}\omega$, for any $\omega \in H^1(\Omega;\mathbb{R}^2)$.

(6) Also from Step (3), $-\Pi^2_{\tilde{r},h}S_1\sigma = -\Pi^2_{\tilde{r},h}S_1\text{curl}\tilde{W}_h\tau - \Pi^2_{\tilde{r},h}S_1\tilde{\Pi}^1_{\tilde{r}+1,h}\eta$. Applying, in order, Step (5), Theorem 5.21, Theorem 5.19, Step (2), and the fact that $\Pi^2_{\tilde{r},h}\Pi^2_{\tilde{r},h} = \Pi^2_{\tilde{r},h}$, we have

$$\begin{aligned}-\Pi^2_{\tilde{r},h}S_1\text{curl}\tilde{W}_h\tau &= \Pi^2_{\tilde{r},h}\text{div}\Pi^{1,-}_{\tilde{r}+1,h}\tilde{W}_h\tau = \Pi^2_{\tilde{r},h}\text{div}\Pi^{1,-}_{\tilde{r}+1,h}\tau = \Pi^2_{\tilde{r},h}\text{div}\tau\\ &= \Pi^2_{\tilde{r},h}(\omega + \Pi^2_{\tilde{r},h}S_1\tilde{\Pi}^1_{\tilde{r}+1,h}\eta) = \omega + \Pi^2_{\tilde{r},h}S_1\tilde{\Pi}^1_{\tilde{r}+1,h}\eta.\end{aligned}$$

Combining, we have $-\Pi^2_{\tilde{r},h}S_1\sigma = \omega$.

(7) Finally, we prove the norm bound. From the boundedness of S_1 in L^2, Theorem 5.19, and Step (1), we obtain

$$\begin{aligned}\|\Pi^2_{\tilde{r},h}S_1\tilde{\Pi}^1_{\tilde{r}+1,h}\eta\|_{L^2(\Omega)} &\leq c\|S_1\tilde{\Pi}^1_{\tilde{r}+1,h}\eta\|_{L^2(\Omega)} \leq c\|\tilde{\Pi}^1_{\tilde{r}+1,h}\eta\|_{L^2(\Omega)}\\ &\leq c\,\|\eta\|_{H^1(\Omega)} \leq c\|\mu\|_{L^2(\Omega)}.\end{aligned}$$

Combining the result with the bound in Step (2), we get $\|\tau\|_{H^1(\Omega)} \leq c(\|\omega\|_{L^2(\Omega)} + \|\mu\|_{L^2(\Omega)})$. Theorem 5.21 then yields

$$\|\text{curl}\tilde{W}_h\tau\|_{L^2(\Omega)} \leq c\|\tau\|_{H^1(\Omega)} \leq c(\|\omega\|_{L^2(\Omega)} + \|\mu\|_{L^2(\Omega)}).$$

From Theorem 5.19 and the bound in Step (1),

$$\|\tilde{\Pi}^1_{,\tilde{r}+1,h}\eta\|_{L^2(\Omega)} \leq c\|\eta\|_{H^1(\Omega)} \leq c\|\mu\|_{L^2(\Omega)}.$$

In view of the definition of σ, these two last bounds imply that $\|\sigma\|_{L^2(\Omega)} \leq c(\|\omega\|_{L^2(\Omega)}+\|\mu\|_{L^2(\Omega)})$, while $\|\text{div}\sigma\|_{L^2(\Omega)} = \|\mu\|_{L^2(\Omega)}$ by Step (4). This finishes the proof. $\square$

Theorem 6.3. *There exists $c > 0$ such that, for solution (σ, u, p) of elasticity system (1.8), and corresponding solution (σ_h, u_h, p_h) of discrete system (4.1), we have*

$$\begin{aligned}&\|\sigma-\sigma_h\|_{H(div,\Omega)}+\|u-u_h\|_{L^2(\Omega)}+\|p-p_h\|_{L^2(\Omega)}\\ \leq &c\inf[\|\sigma-\tau\|_{H(div,\Omega)}+\|u-v\|_{L^2(\Omega)}+\|p-q\|_{L^2(\Omega)}],\end{aligned}$$

where the infimum is taken over all $\tau \in \mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h, \mathbb{R}^2), v \in \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h, \mathbb{R}^2)$, and $q \in \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h)$.

Proof. We need to show that conditions (4.3) and (4.4) are satisfied. Condition (4.3) follows from the fact that, by construction, $\mathrm{div}\mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h, \mathbb{R}^2) \subset \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h; \mathbb{R}^2)$, and the fact that A is coercive. Condition (4.4) comes from Lemma 6.2. This finishes the proof. □

7. Curvilinear Meshes

In practice, meshes generated by CAD software are usually curvilinear. In the following sections, we generalize our result for affine meshes - Theorem 6.3 to curvilinear meshes in the sense of asymptotic h-stability.

7.1. Mesh regularity assumptions.

Definition 7.1. (Curved triangle) A closed set $T \subset \mathbb{R}^2$ is a curved triangle if there exists a C^1-diffeomorphism G_T from reference triangle $\hat{T}$ onto T. This means that G_T is a bijection from $\hat{T}$ to T such that $G_T \in C^1(\hat{T})$ and $G_T^{-1} \in C^1(T)$. We assume additionally that $\det(DG_T(\hat{\mathbf{x}})) > 0$ for any $\hat{\mathbf{x}} \in \hat{T}$.

We represent G_T in the form

$$G_T = \tilde{G}_T + \Phi_T, \tag{7.1}$$

where $\tilde{G}_T : \hat{\boldsymbol{x}} \to B_T\hat{\boldsymbol{x}} + b_T$, $B_T = DG_T(\hat{\boldsymbol{p}})$ with $\hat{\boldsymbol{p}}$ being the centroid of $\hat{T}$, and Φ_T a C^1-mapping from $\hat{T}$ into $\mathbb{R}^2$. The images of edges and vertices of $\hat{T}$ by G_T are edges and vertices of T respectively. We denote $\triangle_i(T) = G_T(\triangle_i(\hat{T})), i = 0, 1, 2,$ and $\triangle(T) = G_T(\triangle(\hat{T}))$.

Definition 7.1 is practically identical with the definition of curved finite elements introduced in [7].

Definition 7.2. A curved triangle T is of class $C^k, k \geq 1$, if the mapping $G_T \in C^k(\hat{T})$. Similarly, a curved triangle T is of class $C^{k,1}, k \geq 1$, if the mapping $G_T \in C^{k,1}(\hat{T})$.

We define $\mathcal{T}_h$ to be a finite set of curved triangles T, where h denotes the maximal distance between two vertices of $T \in \mathcal{T}_h$. We define vertices of $\mathcal{T}_h$ to be vertices of $T \in \mathcal{T}_h$, and we define curves of $\mathcal{T}_h$ to be edges of $T \in \mathcal{T}_h$. We assume that any edge of $T \in \mathcal{T}_h$ is either an edge of another curved triangle in $\mathcal{T}_h$, or part of the boundary of $\mathcal{T}_h$.

Each curve of $\mathcal{T}_h$ is parametrized with a map from the reference unit interval into $\mathbb{R}^2$,

$$[0, 1] \ni s \to \boldsymbol{x}_e(s) \in \mathbb{R}^2$$

The parametrization determines the orientation of the curve.

Let $\zeta(s)$ be the local parametrization for a particular edge of a curved triangle $T \in \mathcal{T}_h$, occupied by a curve e of $\mathcal{T}_h$. This means that $\zeta(s)$ is an affine mapping from the reference interval onto an edge of the reference triangle $\hat{T}$, whose image under the mapping G_T is exactly the particular edge of T. We can choose $\zeta(s)$ so that $G_T(\zeta(s))$ has the same orientation as $\mathbf{x}_e(s)$.

Definition 7.3. (C^0-compatible mesh) $\mathcal{T}_h$ is called a C^0-compatible mesh if, for any curve e and any curved triangle T which contains e as an edge, there is a local parametrization $\zeta(s)$ of e satisfying

$$G_T(\zeta(s)) = \mathbf{x}_e(s).$$

The concept is illustrated in Fig 1.

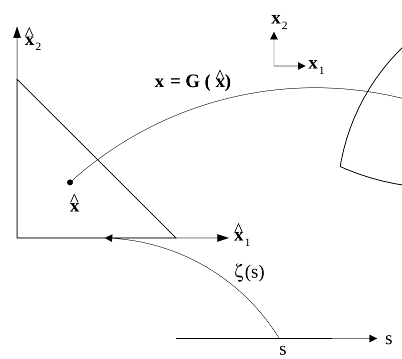


Figure 1. Compatibility of edge and triangle parametrizations.

We denote

$$c_h := \sup_{T\in\mathcal{T}_h}((\sup_{\hat{\mathbf{x}}\in\hat{T}} \|D\Phi_T(\hat{\mathbf{x}})\|)\|B_T^{-1}\|) \tag{7.2}$$

For each $T \in \mathcal{T}_h$, we define $\tilde{T} = \tilde{G}_T(\hat{T})$. We denote by $\tilde{h}_T$ the diameter of $\tilde{T}$ and by $\tilde{\rho}_T$ the diameter of the sphere inscribed in $\tilde{T}$.

We define $\triangle_i(\mathcal{T}_h) = \bigcup_{T\in\mathcal{T}_h} \triangle_i T$, and $\triangle(\mathcal{T}_h) = \bigcup_{T\in\mathcal{T}_h} \triangle T$.

Remark 7.4. In order to simplify analysis, compared with [7], our definition of (7.2) replaces $\|D\Phi_T \cdot B_T^{-1}\|$ with the upper bound $\|D\Phi_T\| \cdot \|B_T^{-1}\|$.

Definition 7.5. The family $(\mathcal{T}_h)_h$ of C^0-compatible meshes is said to be regular if

$$\sup_h \sup_{T\in\mathcal{T}_h} \tilde{h}_T/\tilde{\rho}_T = \sigma < \infty, \text{ and } \lim_{h\to 0} c_h = 0$$

We show the construction of $(\mathcal{T}_h)_h$ of C^0-compatible meshes in Appendix A.

Lemma 7.6. *There exist $c_1, c_2 > 0$ such that, for any triangle T,*

$$c_1\|B_T\| \cdot \|B_T^{-1}\| \le \tilde{h}_T/\tilde{\rho}_T \le c_2\|B_T\| \cdot \|B_T^{-1}\|,$$

where $\mathbf{x} = B_T\hat{\mathbf{x}} + b_T$ is the affine homeomorphism from $\hat{T}$ to T.

Proof. $\tilde{h}_T/\tilde{\rho}_T \le c_2\|B_T\| \cdot \|B_T^{-1}\|$ comes from the geometric meaning of singular values of matrix B_T. $c_1\|B_T\| \cdot \|B_T^{-1}\| \le \tilde{h}_T/\tilde{\rho}_T$ is a consequence of Theorem 3.1.3 in [9]. □

Lemma 7.7. *Let family $(\mathcal{T}_h)_h$ be regular. Then, for any indices $i, j, k, l \in \{1, 2, 3\}$, we have*

$$\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{\hat{\mathbf{x}}\in\hat{T}} \left|\frac{(B_T)_{ij}(D\Phi_T(\hat{\mathbf{x}}))_{kl}}{\det(B_T)}\right| = 0$$

Proof. For any $T \in \mathcal{T}_h$ and any $\hat{\mathbf{x}} \in \hat{T}$, we have

$$\left|\frac{(B_T)_{ij}(D\Phi_T(\hat{\mathbf{x}}))_{kl}}{\det(B_T)}\right| = |(D\Phi_T(\hat{\mathbf{x}}))_{kl}| \|B_T^{-1}\| \cdot |(B_T)_{ij}/\det(B_T)| \frac{1}{\|B_T^{-1}\|}.$$

Since $\|B_T^{-1}\| \cdot \|B_T\| \geq 1$, $\dfrac{1}{\|B_T^{-1}\|} \leq \|B_T\|$. Consequently,

$$\left|\frac{(B_T)_{ij}(D\Phi_T(\hat{\mathbf{x}}))_{kl}}{\det(B_T)}\right| \leq (\|D\Phi_T(\hat{\mathbf{x}})\| \cdot \|B_T^{-1}\|)\|B_T\|^2/|\det(B_T)|.$$

Since $\sup_h \sup_{T\in\mathcal{T}_h} \tilde{h}_T/\tilde{\rho}_T = \sigma < \infty$, $\|B_T\|^2/|\det(B_T)| \leq c\sigma^2$ with $c > 0$.

Since $\lim_{h\to 0} c_h = 0$, we have $\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{\hat{\mathbf{x}}\in\hat{T}} \left|\dfrac{(B_T)_{ij}(D\Phi_T(\hat{\mathbf{x}}))_{kl}}{\det(B_T)}\right| = 0$. □

Lemma 7.8. *If a family $(\mathcal{T}_h)_h$ is regular, then we have*

$$\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{\hat{\mathbf{x}}\in\hat{T}} \|B_T(DG_T(\hat{\mathbf{x}}))^{-1} - I\| = \lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{\hat{\mathbf{x}}\in\hat{T}} \|(DG_T(\hat{\mathbf{x}}))^{-1}B_T - I\| = 0.$$

Proof. For any $T \in \mathcal{T}_h$ and any $\hat{\mathbf{x}} \in \hat{T}$, we have

$$\|B_T(DG_T(\hat{\mathbf{x}}))^{-1} - I\| = \|B_T(B_T + D\Phi_T(\hat{\mathbf{x}}))^{-1} - I\| = \|(I + D\Phi_T(\hat{\mathbf{x}})B_T^{-1})^{-1} - I\|.$$

Since $c_h \to 0$ as $h \to 0$, we have $\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{\hat{\mathbf{x}}\in\hat{T}} \|B_T(DG_T(\hat{\mathbf{x}}))^{-1} - I\| = 0$. Proof of the second property is fully analogous. □

Lemma 7.9. *If a family $(\mathcal{T}_h)_h$ is regular, then we have*

$$\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{\hat{\mathbf{x}}\in\hat{T}} |\det(DG_T(\hat{\mathbf{x}})B_T^{-1}) - 1| = \lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{\hat{\mathbf{x}}\in\hat{T}} |\det(B_T(DG_T(\hat{\mathbf{x}}))^{-1}) - 1| = 0.$$

Proof. Since $\lim_{h\to 0} c_h = 0$, $\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{\hat{\mathbf{x}}\in\hat{T}} |\det(DG_T(\hat{\mathbf{x}})B_T^{-1}) - 1| = 0$. By Lemma 7.8, $\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{\hat{\mathbf{x}}\in\hat{T}} |\det(B_T(DG_T(\hat{\mathbf{x}}))^{-1}) - 1| = 0$. □

8. Finite element spaces on curvilinear meshes

We begin by introducing the relevant finite element spaces on any curved triangle T by the pull back mappings associated with the inverse of G_T, where G_T maps from the reference triangle $\hat{T}$ to T, see Definition 7.1. Then, we define the finite element spaces on a whole mesh $\mathcal{T}_h$ by "gluing" the finite element spaces on curved triangles.

8.1. **Finite element spaces on a curved triangle.** Let T be a curved triangle from Definition 7.1 with G_T denoting the corresponding C^1-diffeomorphism from $\hat{T}$ to T, $\mathbf{x} = G_T(\hat{\mathbf{x}})$. We begin by introducing formally the mapping $\tilde{r}$ from $\triangle(T)$ to $\mathbb{Z}_+$ specifying the local order of discretization.

Definition 8.1. We denote by $\tilde{r}$ a mapping from $\triangle(T)$ to $\mathbb{Z}_+$ such that if $e, f \in \triangle(T)$ and $e \subset f$ then $\tilde{r}(e) \leq \tilde{r}(f)$. With the same symbol $\tilde{r}$ we denote the corresponding mapping from $\triangle(\hat{T})$ to $\mathbb{Z}_+$, $\tilde{r}(\hat{f}) := \tilde{r}(f)$ for any $\hat{f} \in \triangle(\hat{T})$, where $f = G_T(\hat{f})$.

We define now the following FE spaces on T:

Definition 8.2.

$$
\begin{aligned}
\mathcal{P}_{\tilde r}\Lambda^0(T) &:= \{u(\mathbf{x}) : \hat u(\hat{\mathbf{x}}) \in \mathcal{P}_{\tilde r}\Lambda^0(\hat T) \text{ where } u(\mathbf{x}) = \hat u(\hat{\mathbf{x}})\},\\
\mathcal{P}_{\tilde r}\Lambda^1(T) &:= \{\omega(\mathbf{x}) : \hat\omega(\hat{\mathbf{x}}) \in \mathcal{P}_{\tilde r}\Lambda^1(\hat T) \text{ where } \omega(\mathbf{x}) = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))} DG_T(\hat{\mathbf{x}})\hat\omega(\hat{\mathbf{x}})\},\\
\mathcal{P}^-_{\tilde r}\Lambda^1(T) &:= \{\omega(\mathbf{x}) : \hat\omega(\hat{\mathbf{x}}) \in \mathcal{P}^-_{\tilde r}\Lambda^1(\hat T) \text{ where } \omega(\mathbf{x}) = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))} DG_T(\hat{\mathbf{x}})\hat\omega(\hat{\mathbf{x}})\},\\
\mathcal{P}_{\tilde r}\Lambda^2(T) &:= \{u(\mathbf{x}) : \hat u(\hat{\mathbf{x}}) \in \mathcal{P}_{\tilde r}\Lambda^2(\hat T) \text{ where } u(\mathbf{x}) = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))}\hat u(\hat{\mathbf{x}})\},\\
\mathcal{P}_{\tilde r}\Lambda^0(T;\mathbb{R}^2) &:= \{(u_1,u_2) : u_1,u_2 \in \mathcal{P}_{\tilde r}\Lambda^0(T)\},\\
\mathcal{P}_{\tilde r}\Lambda^2(T;\mathbb{R}^2) &:= \{(u_1,u_2) : u_1,u_2 \in \mathcal{P}_{\tilde r}\Lambda^2(T)\},\\
\mathcal{P}_{\tilde r}\Lambda^1(T;\mathbb{R}^2) &:= \left\{ \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} : (\sigma_{11},\sigma_{12})^\top, (\sigma_{21},\sigma_{22})^\top \in \mathcal{P}_{\tilde r}\Lambda^1(T) \right\}.
\end{aligned}
$$

Remark 8.3. Since $G_T : \hat T \to T$ is a C^1-diffeomorphism with $\det(DG_T(\hat{\mathbf{x}})) \neq 0$, for any $\hat{\mathbf{x}} \in \hat T$, the formulae in Definition 8.2 are well-defined. The mappings used in the definition are the standard pull back mappings for differential forms $\Lambda^0, \Lambda^1, \Lambda^2$, see e.g. formulas (2.24), (2.26), (2.27) in [13].

Lemma 8.4. *For any edge $e \in \triangle_1(T)$, let $\zeta(s)$ be the local parametrization for e discussed in Definition 7.3, i.e. the affine mapping from the reference interval onto $\hat e \in \triangle_1(\hat T)$. We have then,*

$$
\begin{aligned}
\mathcal{P}_{\tilde r}\Lambda^0(T)|_e &= \{u(\mathbf{x}) \text{ where } \mathbf{x} \in e : u(G_T(\zeta(s))) \in \mathcal{P}_{\tilde r(\hat e)}(\hat e)\},\\
\mathcal{P}_{\tilde r}\Lambda^1(T)\cdot \mathbf{n}|_e &= \{u(\mathbf{x}) \text{ where } \mathbf{x} \in e : u(G_T(\zeta(s)))\|D(G_T\circ\zeta)(s)\| \ \in \mathcal{P}_{\tilde r(\hat e)}(\hat e)\},\\
\mathcal{P}^-_{\tilde r}\Lambda^1(T)\cdot \mathbf{n}|_e &= \{u(\mathbf{x}) \text{ where } \mathbf{x} \in e : u(G_T(\zeta(s)))\|D(G_T\circ\zeta)(s)\| \ \in \mathcal{P}_{\tilde r(\hat e)-1}(\hat e)\}.
\end{aligned}
$$

In addition, the above equalities do not depend on the choice of the orientation of the local parametrization $\zeta(s)$.

Proof. These are trivial observations on pull back mappings and their restrictions to edges. $\square$

8.2. **Finite element spaces on a C^0-compatible mesh.** Let $\mathcal{T}_h$ be a C^0-compatible mesh from Definition 7.3. We extend the $\tilde r$ mapping to a global map defined on $\triangle(\mathcal{T}_h)$ with values in $\mathbb{Z}_+$ such that if $e \subset f$, then $\tilde r(e) \leq \tilde r(f)$.

Definition 8.5. We put $\Omega_h := \bigcup_{T\in\mathcal{T}_h} T$.

$$
\begin{aligned}
C\Lambda^0(\mathcal{T}_h) &:= \{u \in H^1(\Omega_h) : u \text{ is piece-wise smooth with respect to } \mathcal{T}_h\},\\
C\Lambda^1(\mathcal{T}_h) &:= \{\omega \in H(\mathrm{div},\Omega_h) : \omega \text{ is piece-wise smooth with respect to } \mathcal{T}_h\},\\
C\Lambda^2(\mathcal{T}_h) &:= \{u \in L^2(\Omega_h) : u \text{ is piece-wise smooth with respect to } \mathcal{T}_h\}.
\end{aligned}
$$

We define

$$
\begin{aligned}
\mathcal{P}_{\tilde{r}}\Lambda^0(\mathcal{T}_h) &:= \{u \in C\Lambda^0(\mathcal{T}_h) : u|_T \in \mathcal{P}_{\tilde{r}}\Lambda^0(T), \forall T \in \mathcal{T}_h\},\\
\mathcal{P}_{\tilde{r}}\Lambda^1(\mathcal{T}_h) &:= \{\omega \in C\Lambda^1(\mathcal{T}_h) : \omega|_T \in \mathcal{P}_{\tilde{r}}\Lambda^1(T), \forall T \in \mathcal{T}_h\},\\
\mathcal{P}^-_{\tilde{r}}\Lambda^1(\mathcal{T}_h) &:= \{\omega \in C\Lambda^1(\mathcal{T}_h) : \omega|_T \in \mathcal{P}^-_{\tilde{r}}\Lambda^1(T), \forall T \in \mathcal{T}_h\},\\
\mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h) &:= \{u \in C\Lambda^2(\mathcal{T}_h) : u|_T \in \mathcal{P}_{\tilde{r}}\Lambda^2(T), \forall T \in \mathcal{T}_h\},\\
\mathcal{P}_{\tilde{r}}\Lambda^0(\mathcal{T}_h;\mathbb{R}^2) &:= [\mathcal{P}_{\tilde{r}}\Lambda^0(\mathcal{T}_h)]^2, \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h;\mathbb{R}^2) := [\mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h)]^2,\\
\mathcal{P}_{\tilde{r}}\Lambda^1(\mathcal{T}_h;\mathbb{R}^2) &:= \left\{ \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} : (\sigma_{11},\sigma_{12})^\top, (\sigma_{21},\sigma_{22})^\top \in \mathcal{P}_{\tilde{r}}\Lambda^1(\mathcal{T}_h,\mathbb{R}^2) \right\}.
\end{aligned}
$$

Remark 8.6. According to Lemma 8.4 and the fact that $\mathcal{T}_h$ is C^0-compatible, we can conclude that $\mathcal{P}_{\tilde{r}}\Lambda^0(\mathcal{T}_h)|_T = \mathcal{P}_{\tilde{r}}\Lambda^0(T), \mathcal{P}_{\tilde{r}}\Lambda^1(\mathcal{T}_h)|_T = \mathcal{P}_{\tilde{r}}\Lambda^1(T), \mathcal{P}^-_{\tilde{r}}\Lambda^1(\mathcal{T}_h)|_T = \mathcal{P}^-_{\tilde{r}}\Lambda^1(T), \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h)|_T = \mathcal{P}_{\tilde{r}}\Lambda^2(T)$, for any $T \in \mathcal{T}_h$. For standard (not curved) triangulations $\mathcal{T}_h$, spaces $\mathcal{P}_{\tilde{r}}\Lambda^0(\mathcal{T}_h), \mathcal{P}_{\tilde{r}}\Lambda^1(\mathcal{T}_h), \mathcal{P}^-_{\tilde{r}}\Lambda^1(\mathcal{T}_h), \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h)$ coincide with those analyzed in [19].

Lemma 8.7.

$$
\begin{aligned}
&\mathcal{P}_{\tilde{r}}\Lambda^1(\mathcal{T}_h) \subset \mathcal{P}^-_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h) \subset \mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h),\\
&div\mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h) \subset \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h), curl\mathcal{P}_{\tilde{r}+1}\Lambda^0(\mathcal{T}_h) \subset \mathcal{P}_{\tilde{r}}\Lambda^1(\mathcal{T}_h).
\end{aligned}
$$

Proof. The proof is straightforward. □

9. Preliminaries for the proof of stability for curvilinear meshes

We begin by recalling our assumptions on the domain and meshes: Ω is a (curvilinear) polygon and it is meshed with a family $(\mathcal{T}_h)_h$ of C^0-compatible meshes of class $C^{1,1}$. For any mesh $\mathcal{T}_h$, mapping $\tilde{r} : \triangle(\mathcal{T}_h) \to \mathbb{Z}_+$ defines a locally variable order of discretization that satisfies the minimum rule. The maximum order is limited, i.e. $\sup_h \sup_{T\in\mathcal{T}_h} \tilde{r}(T) < \infty$.

As in Section 5, we shall design operators $\Pi^2_{\tilde{r},T}$, $\Pi^{1,-}_{\tilde{r}+1,T}$ and W_T in order to make the left and the right diagrams in (4.5) commute. In the case of affine meshes (Section 5), we define $\Pi^{1,-}_{\tilde{r}+1,T}$ and W_T on physical triangles, then "pull them back" to the reference triangle. We designed $\Pi^{1,-}_{\tilde{r}+1,T}$ and W_T in such a way that their "pull backs" are $\Pi^{1,-}_{\tilde{r}+1,\hat{T},t_{\tilde{r}(\hat{T})}}$ and $C_{t_{\tilde{r}(\hat{T})}}$ respectively (Definition 5.17). In the case of curvilinear meshes, the approach to $\Pi^2_{\tilde{r},T}$ and $\Pi^{1,-}_{\tilde{r}+1,T}$ is similar to that in Section 5. But the treatment to W_T is quite different. In order to make the right diagram in (4.5) commute (we need the commutativity on the physical meshes), we incorporate the construction of $\Pi^{1,-}_{\tilde{r}+1,T}$ into the definition of W_T. Then we use a special mapping such that the "pull back" of W_T is a projection operator to $\mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$. In general, this kind of "pull back" of W_T to the reference triangle does not coincide with $C_{t_{\tilde{r}(T)}}$ from Definition 5.13. This means that, in general, W_T is not necessarily well-defined. But we manage to prove that the "pull back" of W_T will converge to $C_{t_{\tilde{r}(T)}}$ as $h \to 0$. So W_T becomes well-defined when h is small enough. The work presented here illuminates the substantial difference between the stability analysis for curvilinear meshes and that for affine meshes. This is in particular reflected in the proof of Theorem 9.15, which uses heavily Lemmas 7.7,7.8,7.9.

In order to make the paper more readable, most proofs of results discussed in this section have been moved into Appendix C.

Definition 9.1. For any $T \in \mathcal{T}_h$, we define a linear operator $\Pi^2_{\tilde{r},T} : L^2(T) \longrightarrow \mathcal{P}_{\tilde{r}}\Lambda^2(T)$ by the relations

$$\int_T (\Pi^2_{\tilde{r},T} u(\mathbf{x}) - u(\mathbf{x}))\hat{\psi}(\hat{\mathbf{x}}(\mathbf{x}))\, d\mathbf{x} = 0 \quad \forall \hat{\psi} \in \mathcal{P}_{\tilde{r}(T)}(\hat{T}) \tag{9.1}$$

Above, $\hat{\mathbf{x}}(\mathbf{x})$ signifies the inverse of the element map $\mathbf{x} = G_T(\hat{\mathbf{x}})$.

Definition 9.2. Operator $\Pi^2_{\tilde{r},\hat{T}} : L^2(\hat{T}) \longrightarrow \mathcal{P}_{\tilde{r}}\Lambda^2(\hat{T})$ will denote the L^2-projection in the reference space,

$$\int_T (\Pi^2_{\tilde{r},\hat{T}} \hat{u}(\hat{\mathbf{x}}) - \hat{u}(\hat{\mathbf{x}}))\hat{\psi}(\hat{\mathbf{x}})\, d\hat{\mathbf{x}} = 0 \quad \forall \hat{\psi} \in \mathcal{P}_{\tilde{r}(T)}(\hat{T}) \tag{9.2}$$

Remark 9.3. Operator $\Pi^2_{\tilde{r},T}$ is a weighted L^2-projection in the physical space. For a regular triangle (affine element map), the jacobian is constant, and $\Pi^2_{\tilde{r},T}$ reduces to the standard L^2-projection in the physical space.

Lemma 9.4. *For any $T \in \mathcal{T}_h$, and arbitrary $u(\mathbf{x}) \in L^2(T)$, we define $\hat{u}(\hat{\mathbf{x}})$ by the relation:*

$$u(\mathbf{x}(\hat{\mathbf{x}})) = \frac{\hat{u}(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}}))}$$

Then $\hat{u}(\hat{\mathbf{x}}) \in L^2(\hat{T})$, and

$$\Pi^2_{\tilde{r},T} u(\mathbf{x}(\hat{\mathbf{x}})) = \frac{\Pi^2_{\tilde{r},\hat{T}} \hat{u}(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}})))}$$

Above, $\mathbf{x}(\hat{\mathbf{x}})$ signifies the element map $\mathbf{x} = G_T(\hat{\mathbf{x}})$.

Proof. Proof follows immediately from Lemma B.6 and the definitions of the two projections. □

Lemma 9.5. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$,*

$$\|\Pi^2_{\tilde{r},T} u - P_{\tilde{r},T} u\|_{L^2(T)} \le \varepsilon \|u\|_{L^2(T)}, \forall u \in L^2(T).$$

Here $P_{\tilde{r},T}$ is the standard L^2-projection onto $\mathcal{P}_{\tilde{r}}\Lambda^2(T)$.

Proof. Please see Appendix C. □

9.1. Projection Based Interpolation onto $\mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h)$.

Definition 9.6. For any $T \in \mathcal{T}_h$, we define a linear operator $\Pi^1_{\tilde{r}+1,T} : H^1(T;\mathbb{R}^2) \longrightarrow \mathcal{P}_{\tilde{r}+1}\Lambda^1(T)$ by the relations

$$\int_T \operatorname{div}(\Pi^1_{\tilde{r}+1,T}\omega - \omega)(\mathbf{x})\hat{\psi}(\hat{\mathbf{x}}(\mathbf{x}))d\mathbf{x} = 0 \quad \forall \hat{\psi} \in \mathcal{P}_{\tilde{r}(T)}(\hat{T})/\mathbb{R} \tag{9.3}$$

$$\int_T (\Pi^1_{\tilde{r}+1,T}\omega(\mathbf{x}) - \omega(\mathbf{x}))^\top DG_T(\hat{\mathbf{x}})^{-\top} \operatorname{curl}_{\hat{\mathbf{x}}}\hat{\varphi}(\hat{\mathbf{x}}(\mathbf{x}))d\mathbf{x} = 0 \quad \forall \hat{\varphi} \in \mathring{\mathcal{P}}_{\tilde{r}(T)+2}(\hat{T}) \tag{9.4}$$

(9.5)

$$\int_{[0,1]} [(\Pi^1_{\tilde{r}+1,T}\omega - \omega)(\mathbf{x}_e(s)) \cdot \mathbf{n}(\mathbf{x}_e(s))]\hat{\eta}(s)\|\dot{\mathbf{x}}_e(s)\|ds = 0 \quad \forall \hat{\eta} \in \mathcal{P}_{\tilde{r}(e)+1}([0,1]) \forall e \in \triangle_1(T)$$

Here $\mathbf{x} = G_T(\hat{\mathbf{x}})$ for any $\hat{\mathbf{x}} \in \hat{T}$, $\mathbf{x}_e(s) : [0,1] \to e$ is the parametrization of e, and $\mathbf{n}$ is a unit normal vector along e (the choice of its direction does not matter).

Definition 9.7. (Projection Based Interpolation operator onto $\mathcal{P}_{\tilde{r}+1}\Lambda^1(\hat{T})$) We define a linear operator $\Pi^1_{\tilde{r}+1,\hat{T}} : H(\hat{T}) \longrightarrow \mathcal{P}_{\tilde{r}+1}\Lambda^1(\hat{T})$ by the relations

$$\int_{\hat{T}} \mathrm{div}_{\hat{\mathbf{x}}}(\Pi^1_{\tilde{r}+1,\hat{T}}\hat{\omega} - \hat{\omega})\hat{\psi} d\hat{\mathbf{x}} = 0 \quad \forall \hat{\psi} \in \mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T})/\mathbb{R} \tag{9.6}$$

$$\int_{\hat{T}} (\Pi^1_{\tilde{r}+1,\hat{T}}\hat{\omega} - \hat{\omega}) \cdot \mathrm{curl}_{\hat{\mathbf{x}}}\hat{\varphi} d\hat{\mathbf{x}} = 0 \quad \forall \hat{\varphi} \in \mathring{\mathcal{P}}_{\tilde{r}(\hat{T})+2}(\hat{T}) \tag{9.7}$$

$$\int_{\hat{e}} (\Pi^1_{\tilde{r}+1,\hat{T}}\hat{\omega} - \hat{\omega}) \cdot \hat{\mathbf{n}}\hat{\eta} d\hat{s} = 0 \quad \forall \hat{\eta} \in \mathcal{P}_{\tilde{r}(\hat{e})+1}(\hat{e}) \, \forall \hat{e} \in \triangle_1(\hat{T}) \tag{9.8}$$

Remark 9.8. The operator $\Pi^1_{\tilde{r}+1,\hat{T}}$ is the Projection Based Interpolation operator onto $\mathcal{P}_{\tilde{r}+1}\Lambda^1(\hat{T})$ defined in [19]. The operator $\Pi^1_{\tilde{r}+1,T}$ is defined by the pull back mapping from $\hat{T}$ to T.

Lemma 9.9. *For any $T \in \mathcal{T}_h$, any $\omega \in [H^1(T)]^2$, we define $\hat{\omega}$ by the relation*

$$\omega(\mathbf{x}(\hat{\mathbf{x}})) = \frac{DG_T(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}}))}\hat{\omega}(\hat{x})$$

Then $\hat{\omega}(\hat{\mathbf{x}}) \in H(\hat{T})$, and

$$\Pi^1_{\tilde{r}+1,T}\omega(\mathbf{x}(\hat{\mathbf{x}})) = \frac{DG_T(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}}))}\Pi^1_{\tilde{r}+1,\hat{T}}\hat{\omega}(\hat{\mathbf{x}})$$

Proof. Please see Appendix C. □

Lemma 9.10. *For any $T \in \mathcal{T}_h$, and any $\omega \in [H^1(T)]^2$, we have $\Pi^2_{\tilde{r},T} div\omega = div\Pi^1_{\tilde{r}+1,T}\omega$.*

Proof. Please see Appendix C. □

Lemma 9.11. *There exists $\delta > 0$ and $C > 0$ such that, for any $h < \delta$, we have*

$$\|\Pi^1_{\tilde{r},T}\omega\|_{L^2(T)} \leq C\|\omega\|_{H^1(T)} \quad \forall T \in \mathcal{T}_h, \omega \in H^1(T;\mathbb{R}^2)$$

For affine meshes, the inequality above holds for any $h > 0$.

Proof. Please see Appendix C. □

9.2. **Modified Projection Based Interpolation onto $\mathcal{P}^-_{\tilde{r}+1}\Lambda^1(T)$ and modified operator W onto $\mathcal{P}_{\tilde{r}+2}\Lambda^0(T)$.**

Definition 9.12. (PB interpolation operator onto $\mathcal{P}^-_{\tilde{r}+1}\Lambda^1(T)$) For $t_{\tilde{r}(\hat{T})}$ given in Definition 5.17, and for any $T \in \mathcal{T}_h$, we define a linear operator $\Pi^{1,-}_{\tilde{r}+1,T}$: $H^1(T;\mathbb{R}^2) \longrightarrow \mathcal{P}^-_{\tilde{r}+1}\Lambda^1(T)$ by the relations

$$\int_T \mathrm{div}(\Pi^{1,-}_{\tilde{r}+1,T}\omega - \omega)(\mathbf{x})\hat{\psi}(\hat{\mathbf{x}}(\mathbf{x}))\, d\mathbf{x} = 0 \quad \forall \hat{\psi} \in \mathcal{P}_{\tilde{r}(T)}(\hat{T})/\mathbb{R} \tag{9.9}$$

$$\int_T (\Pi^{1,-}_{\tilde{r}+1,T}\omega(\mathbf{x}) - \omega(\mathbf{x}))^\top DG_T(\hat{\mathbf{x}}(\mathbf{x}))^{-\top}\hat{\mathbf{h}}_i(\hat{\mathbf{x}}(\mathbf{x}), t_{\tilde{r}(\hat{T})})d\mathbf{x} = 0 \quad 1 \leq i \leq k_{\tilde{r}} \tag{9.10}$$

(9.11)
$$\int_{[0,1]}[(\Pi^{1,-}_{\tilde{r}+1,T}\omega-\omega)(\mathbf{x}_e(s))\cdot\mathbf{n}(\mathbf{x}_e(s))]\hat{\eta}(s)\|\dot{\mathbf{x}}_e(s)\|ds=0 \quad \forall\hat{\eta}\in\mathcal{P}_{\tilde{r}(e)}([0,1]),\forall e\in\triangle_1(T)$$

In the above, $\hat{\mathbf{x}}=\hat{\mathbf{x}}(\mathbf{x})$ signifies the inverse of the element map.

Theorem 9.13. *The operator $\Pi^{1,-}_{\tilde{r}+1,T}:H^1(T;\mathbb{R}^2)\longrightarrow\mathcal{P}^-_{\tilde{r}+1}\Lambda^1(T)$ is well-defined, and we have the following result.*

$$div\Pi^{1,-}_{\tilde{r}+1,T}\omega=\Pi^2_{\tilde{r},T}div\omega \quad \forall\omega\in H^1(T;\mathbb{R}^2).$$

There exist $\delta>0$ and $C>0$ such that, for $h\le\delta$, $T\in\mathcal{T}_h$, and $\omega\in H^1(T;\mathbb{R}^2)$,

$$\|\Pi^{1,-}_{\tilde{r}+1,T}\omega\|_{L^2(T)}\le C\|\omega\|_{H^1(T)}$$

Proof. The proof is analogous to that of Lemma 9.9, Lemma 9.10, and Lemma 9.11. □

Definition 9.14. For $t_{\tilde{r}(\hat{T})}$ given in Definition 5.17, and for any $T\in\mathcal{T}_h$, we define a linear operator $W_T:H^1(T;\mathbb{R}^2)\longrightarrow\mathcal{P}_{\tilde{r}+2}\Lambda^0(T;\mathbb{R}^2)$ by the following relations

$$\int_T \operatorname{div}(W_T\omega-\omega)(\mathbf{x})\hat{\psi}(\hat{\mathbf{x}}(\mathbf{x}))d\mathbf{x}=0 \quad \forall\hat{\psi}\in\mathcal{P}_{\tilde{r}(T)}(\hat{T})/\mathbb{R} \tag{9.12}$$

$$\int_T (W_T\omega(\mathbf{x})-\omega(\mathbf{x}))^\top DG_T(\hat{\mathbf{x}}(\mathbf{x}))^{-\top}\hat{\mathbf{h}}_i(\hat{\mathbf{x}}(\mathbf{x}),t_{\tilde{r}(\hat{T})})d\mathbf{x}=0 \quad 1\le i\le k_{\tilde{r}} \tag{9.13}$$

(9.14)
$$\int_{[0,1]}[(W_{T,t}\omega-\omega)(\mathbf{x}_e(s))\cdot\mathbf{n}(\mathbf{x}_e(s))]\hat{\eta}(s)\|\dot{\mathbf{x}}_e(s)\|ds=0 \quad \forall\hat{\eta}\in\mathcal{P}_{\tilde{r}(e)}([0,1]),\forall e\in\triangle_1(T)$$

(9.15)
$$\int_{[0,1]}[(W_{T,t}\omega-\omega)(\mathbf{x}_e(s))\cdot\mathbf{t}(\mathbf{x}_e(s))]\hat{\eta}(s)\|\dot{\mathbf{x}}_e(s)\|ds=0 \quad \forall\hat{\eta}\in\mathcal{P}_{\tilde{r}(e)}([0,1]),\forall e\in\triangle_1(T)$$

$$W_{T,t}\omega=0 \text{ at all vertices of } T \tag{9.16}$$

Here $\mathbf{n},\mathbf{t}$ denote the normal and tangent unit vectors along ∂T.

Theorem 9.15. *There exist $\delta>0$ and $C>0$ such that, for any $h<\delta$, $T\in\mathcal{T}_h$, operator $W_T:H^1(T;\mathbb{R}^2)\longrightarrow\mathcal{P}_{\tilde{r}+2}\Lambda^0(T;\mathbb{R}^2)$ is well-defined and,*

$$\|curlW_T\omega\|_{L^2(T)}\le C(\tilde{h}_T^{-1}\|\omega\|_{L^2(T)}+\|\omega\|_{H^1(T)}) \quad \forall\omega\in H^1(T;\mathbb{R}^2)$$

For affine meshes, W_T is well-defined and the above inequality holds for any $h>0$.

Proof. Please see Appendix C. □

9.3. **Projection operators on the whole curvilinear meshes.**

Definition 9.16. Similar to Definition 5.18, We define the following global interpolation operators,

$$\Pi^2_{\tilde{r},\mathcal{T}_h}:L^2(\Omega)\longrightarrow\mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h), \quad (\Pi^2_{\tilde{r},\mathcal{T}_h}u)|_T=\Pi^2_{\tilde{r},T}(u|_T)$$

$$\tilde{\Pi}^2_{\tilde{r},\mathcal{T}_h}:L^2(\Omega;\mathbb{R}^2)\longrightarrow\mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h;\mathbb{R}^2), \quad (\tilde{\Pi}^2_{\tilde{r},\mathcal{T}_h}(u_1,u_2)^\top)|_T=(\Pi^2_{\tilde{r},T}(u_1|_T),\Pi^2_{\tilde{r},T}(u_2|_T))^\top$$

$$\Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}:H^1(\Omega;\mathbb{R}^2)\longrightarrow\mathcal{P}^{-1}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h), \quad (\Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}\omega)|_T=\Pi^{1,-}_{\tilde{r}+1,T}(\omega|_T)$$

$$\tilde{\Pi}^1_{\tilde{r}+1,\mathcal{T}_h}:H^1(\Omega;\mathbb{M})\longrightarrow\mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h;\mathbb{R}^2), \quad (\tilde{\Pi}^1_{\tilde{r}+1,\mathcal{T}_h}\sigma)|_T=\begin{bmatrix}\tau_{11} & \tau_{12}\\ \tau_{21} & \tau_{22}\end{bmatrix}$$

where

$$\begin{bmatrix} \tau_{11} \\ \tau_{12} \end{bmatrix} = \Pi^1_{\tilde{r}+1,\mathcal{T}_h} \begin{bmatrix} \sigma_{11}|_T \\ \sigma_{12}|_T \end{bmatrix}, \quad \begin{bmatrix} \tau_{21} \\ \tau_{22} \end{bmatrix} = \Pi^1_{\tilde{r}+1,\mathcal{T}_h} \begin{bmatrix} \sigma_{21}|_T \\ \sigma_{22}|_T \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$$W_{\mathcal{T}_h} : H^1(\Omega;\mathbb{R}^2) \longrightarrow \mathcal{P}_{\tilde{r}+2}\Lambda^0(\mathcal{T}_h;\mathbb{R}^2), \quad (W_{\mathcal{T}_h}\omega)|_T = W_T(\omega|_T)$$

for all $T \in \mathcal{T}_h$.

Remark 9.17. Since $(\mathcal{T}_h)_h$ is C^0-compatible, operators $\Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}$, $\tilde{\Pi}^1_{\tilde{r}+1,\mathcal{T}_h}$ and $W_{\mathcal{T}_h}$ are well-defined.

Theorem 9.18. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $h < \delta$,*

$$\|\Pi^2_{\tilde{r},\mathcal{T}_h} u - P_{\tilde{r},\mathcal{T}_h} u\|_{L^2(\Omega)} \leq \varepsilon \|u\|_{L^2(\Omega)} \quad \forall u \in L^2(\Omega)$$

Here $P_{\tilde{r},\mathcal{T}_h}$ is the standard L^2-projection onto $\mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h)$.

Proof. This is an immediate result of Lemma 9.5. □

Theorem 9.19. *There exist $\delta > 0$ and $C > 0$ such that, for any $h < \delta$, we have*

$$\|\tilde{\Pi}^1_{\tilde{r}+1,\mathcal{T}_h}\sigma\|_{L^2(\Omega)} \leq C\|\sigma\|_{H^1(\Omega)}, \quad \|\Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}\omega\|_{L^2(\Omega)} \leq C\|\omega\|_{H^1(\Omega)},$$

for any $\sigma \in H^1(\Omega;\mathbb{M})$ and $\omega \in H^1(\Omega;\mathbb{R}^2)$. For affine meshes, the inequalities above hold for any $h > 0$. Moreover,

$$div\tilde{\Pi}^1_{\tilde{r}+1,\mathcal{T}_h}\sigma = \tilde{\Pi}^2_{\tilde{r},\mathcal{T}_h} div\sigma, \quad div\Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}\omega = \Pi^2_{\tilde{r},\mathcal{T}_h} div\omega$$

Proof. This is an immediate result of Lemma 9.11 and Theorem 9.13. □

Definition 9.20. Let R_h denote the generalized Clement interpolant operator from Theorem 5.1 in [7], mapping $H^1(\Omega;\mathbb{R}^2)$ into $\mathcal{P}_1\Lambda^0(\mathcal{T}_h;\mathbb{R}^2)$. We define

$$\tilde{W}_h = W_h(I - R_h) + R_h$$

Theorem 9.21. *There exist $\delta > 0$ and $C > 0$ such that, for any $h < \delta$,*

$$\|curl\tilde{W}_h\omega\|_{L^2(\Omega)} \leq C\|\omega\|_{H^1(\Omega)} \quad \forall \omega \in H^1(\Omega;\mathbb{R}^2)$$

For affine meshes, the inequality holds for any $h > 0$. Operator $\tilde{W}_h$ maps $H^1(\Omega;\mathbb{R}^2)$ into $\mathcal{P}_{\tilde{r}+2}\Lambda^0(T;\mathbb{R}^2)$ and satisfies the condition

$$\Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}\omega = \Pi^{1,-}_{\tilde{r}+1,\mathcal{T}_h}\tilde{W}_h\omega \quad \forall \omega \in H^1(\Omega;\mathbb{R}^2)$$

Proof. We utilize Example 2 from [7] with uniform order equal 1 to construct operator R_h. Since $(\mathcal{T}_h)_h$ is C^0-compatible, and $c_h \to 0$ as $h \to 0$, operator R_h maps $H^1(\Omega;\mathbb{R}^2)$ into $\mathcal{P}_1\Lambda^0(T;\mathbb{R}^2) \subset \mathcal{P}_{\tilde{r}+2}\Lambda^0(T;\mathbb{R}^2)$. The the proof will be the same as that of Theorem 5.21. □

10. Asymptotic stability of the finite element discretization on curvilinear meshes

Lemma 10.1. *There exist $\delta > 0$ and $c > 0$ such that, for any $h < \delta$ and any $(\omega,\mu) \in \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h) \times \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h;\mathbb{R}^2)$, there exists $\sigma \in \mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h;\mathbb{R}^2)$ such that*

$$div\sigma = \mu, \quad -\Pi^2_{\tilde{r},\mathcal{T}_h} S_1\sigma = \omega$$

and

$$\|\sigma\|_{H(div,\Omega)} \leq c(\|\mu\|_{L^2(\Omega)} + \|\omega\|_{L^2(\Omega)})$$

Here, the constant c depends on $\sup_h \sup_{T\in\mathcal{T}_h} \tilde{r}(T)$. For affine meshes, the inequality above holds for any $h > 0$.

Proof. The proof is the same as that of Lemma 6.2. □

Theorem 10.2. *There exist $\delta > 0$ and $c > 0$ such that, for solution (σ, u, p) of elasticity system (1.8), and corresponding solution (σ_h, u_h, p_h) of discrete system (4.1), we have*

$$\begin{aligned}&\|\sigma - \sigma_h\|_{H(div,\Omega)} + \|u - u_h\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)}\\ \le & c \inf[\|\sigma - \tau\|_{H(div,\Omega)} + \|u - v\|_{L^2(\Omega)} + \|p - q\|_{L^2(\Omega)}],\end{aligned}$$

where the infimum is taken over all $\tau \in \mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h, \mathbb{R}^2), v \in \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h, \mathbb{R}^2)$, and $q \in \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h)$, for $h < \delta$. For affine meshes, the inequality holds for any $h > 0$.

Proof. We need to show that conditions (4.3) and (4.4) are satisfied asymptotically in h. Condition (4.3) follows from the fact that, by construction, $\mathrm{div}\mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h, \mathbb{R}^2) \subset \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h; \mathbb{R}^2)$, and the fact that A is coercive.

We turn now to condition (4.4). According to Lemma 10.1, there exist $\delta > 0$ and $c > 0$ such that, for $h < \delta$ and $(\omega, \mu) \in \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h) \times \mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h; \mathbb{R}^2)$, there exists $\sigma \in \mathcal{P}_{\tilde{r}+1}\Lambda^1(\mathcal{T}_h; \mathbb{R}^2)$ such that $\mathrm{div}\sigma = \mu$, $-\Pi^2_{\tilde{r},\mathcal{T}_h} S_1\sigma = \omega$, and $\|\sigma\|_{H(\mathrm{div},\Omega)} \le c(\|\mu\|_{L^2(\Omega)} + \|\omega\|_{L^2(\Omega)})$. We have then

$$\begin{aligned}\langle \mathrm{div}\sigma, \mu\rangle - \langle S_1\sigma, \omega\rangle =&\langle \mathrm{div}\sigma, \mu\rangle - \langle \Pi^2_{\tilde{r},\mathcal{T}_h} S_1\sigma, \omega\rangle + \langle (\Pi^2_{\tilde{r},\mathcal{T}_h} - P_{\tilde{r},\mathcal{T}_h}) S_1\sigma, \omega\rangle\\ \ge& c\|\sigma\|_{H(\mathrm{div},\Omega)}(\|\mu\|_{L^2(\Omega)} + \|\omega\|_{L^2(\Omega)}) + \langle (\Pi^2_{\tilde{r},\mathcal{T}_h} - P_{\tilde{r},\mathcal{T}_h}) S_1\sigma, \omega\rangle.\end{aligned}$$

According to Theorem 9.18, for sufficiently small h,

$$|\langle (\Pi^2_{\tilde{r},\mathcal{T}_h} - P_{\tilde{r},\mathcal{T}_h}) S_1\sigma, \omega\rangle| \le \frac{c}{2}\|\sigma\|_{L^2(\Omega)}\|\omega\|_{L^2(\Omega)}$$

So, asymptotically in h, we have

$$\langle \mathrm{div}\sigma, \mu\rangle - \langle S_1\sigma, \omega\rangle \ge \frac{c}{2}\|\sigma\|_{H(\mathrm{div},\Omega)}(\|\mu\|_{L^2(\Omega)} + \|\omega\|_{L^2(\Omega)}).$$

For affine meshes, $\Pi^2_{\tilde{r},\mathcal{T}_h}$ reduces to the standard L^2-projection. The inequality above holds then for any $h > 0$. This finishes the proof. □

11. Numerical experiments

We continue numerical experiments initiated in [19] where we investigated rates of convergence for uniform h-refinements in presence of non-uniform polynomial order p, and tried out p-adaptivity (with no underlying theory at presence). The experiments confirmed the optimal h-convergence rates and indicated the p-convergence as well.

Following [19], we consider the L-shape domain, and use a manufactured solution corresponding to the exact solution of the homogeneous equilibrium equations (zero volume forces), and the corresponding unbounded L-shape domain extending to infinity. The manufactured solution is designed in such a way that both p and all stress components have the same singularity at the origin characterized by term $r^{-0.39596}$ where r is the distance to the origin.

Experiments presented below focus on h-adaptivity for meshes with uniform order p, and the ultimate goal of this research - the hp-adaptivity. We investigate both affine and curvilinear meshes. We use the standard “greedy algorithm” for h-refinements, and the two-grid hp-algorithm for hp-refinements, see [12] for details. The hp-algorithm is based on the standard Projection Based (PB) interpolation.

All convergence plots are displayed on log-log scale, error vs. number of degrees-of-freedom (d.o.f.). The error is always measured in terms of percent of the total norm of the solution.

Uniform h-refinements on affine meshes. We begin with a verification of stability on uniform meshes. Fig. 2 presents the L-shape domain with an initial mesh of six elements. Fig. 3 displays the actual approximation error compared with the

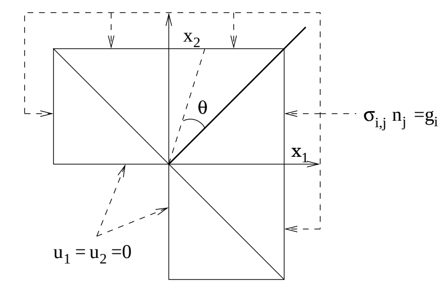


FIGURE 2. The L shape domain with initial mesh.

best approximation error for a sequence of uniformly refined meshes of zero order[2]. As expected, the two curves are parallel to each other. We repeat now the same

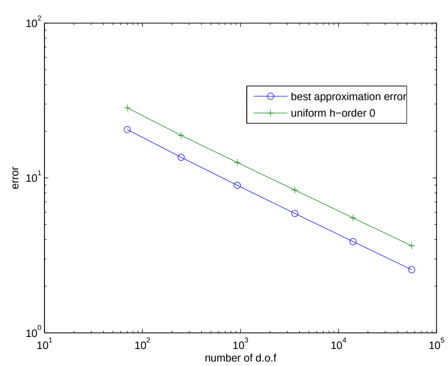


FIGURE 3. Uniform h-refinements for $p = 0$. FE error vs. best approximation error.

experiment starting with an initial mesh of elements with order varying from zero to four, shown in Fig. 4. The corresponding FE error is compared again with the best approximation error in Fig. 5. The two lines are again parallel to each other with the slope determined by the lowest order elements in the mesh (same as in the first example).

[2]We always refer to the order of approximation for the displacement.

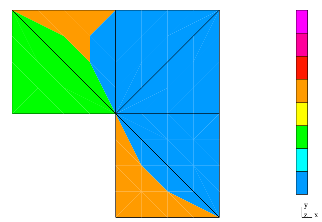


FIGURE 4. The L shape domain with initial mesh.

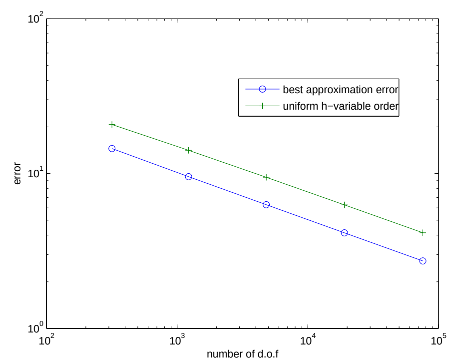


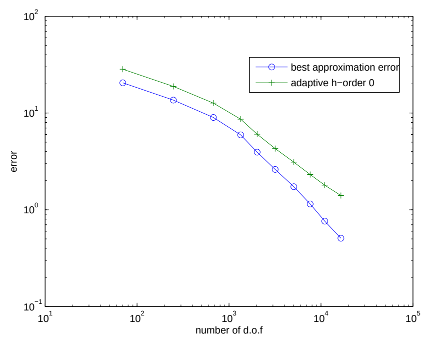
FIGURE 5. Uniform h-refinements for a mesh of variable order. FE error vs. best approximation error.

Adaptive h- and hp-refinements on affine meshes. We continue now the numerical verification of stability for non-uniform meshes resulting from h- and hp-refinements. Unfortunately, we face a slight discrepancy between the presented theory and the numerical experiments as the code is using 1-irregular meshes which we have not accounted for in our theoretical analysis.

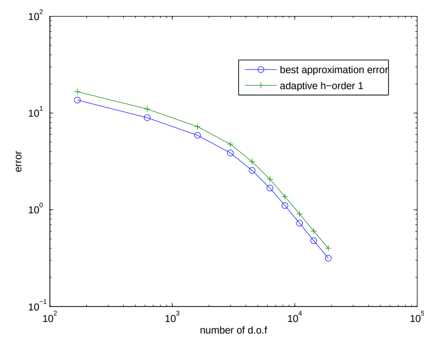
Fig. 6 presents convergence history for adaptive h-refinements for meshes of order $p = 0, 1, 2$ and adaptive hp-refinements starting with the mesh of zero order. The approximation error is compared with the best approximation error computed on the same meshes (generated by the adaptive algorithm). Results for h-refined meshes of order $p = 1, 2$, and the hp-refinements confirm the stability result. The result for the lowest order elements, though, reflects some loss of stability. The only possible explanation that we have at the moment, is the use of meshes with hanging nodes.

Finally, in Fig. 7, we compare the convergence history for all tested refinements. The hp-adaptivity produces the best results although, in the presented range, the rate seems to be still only algebraic.

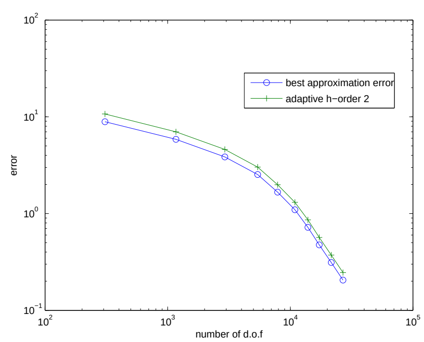
Adaptive h- and hp-refinements on curvilinear meshes. We use the same manufactured solution on the circular L-shape domain shown in Fig. 8. The six triangles in the initial mesh are parametrized using the transfinite interpolation technique, see [12], p. 201, for details. The FE error is compared again with



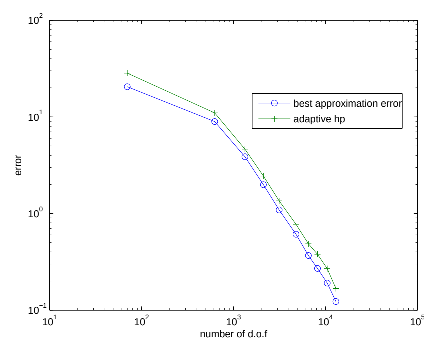
(a) Adaptive h-refinements, $p = 0$.



(b) Adaptive h-refinements, $p = 1$.



(c) Adaptive h-refinements, $p = 2$.



(d) Adaptive hp-refinements.

FIGURE 6. Comparison of FE error for adaptively refined affine meshes with the best approximation error.

the best approximation error for a sequence of h-adaptive meshes and $p = 0, 1, 2$, in Fig. 9. The results indicate again a slight loss of stability for the elements of lowest order. Finally, in Fig. 10, we compare the convergence history for all tested refinements. The hp-adaptivity delivers again the best results but the exponential convergence is questionable. This may indicate that the use of standard Projection-Based interpolation operator in the reference domain is not optimal.

12. CONCLUSIONS

We have presented a complete h-stability analysis for a generalization of Arnold-Falk-Winther elements to curvilinear meshes of variable order in two space dimensions. The stability analysis for both generalizations: variable order elements, and curvilinear elements proved to be rather non-trivial. The case of variable elements has been tackled with a novel logical construction showing the existence of necessary interpolation operators rather than constructing them explicitly. The presented construction departs from operators used by Arnold, Falk and Winther and modifies the Projection-Based (PB) interpolation operators as well.

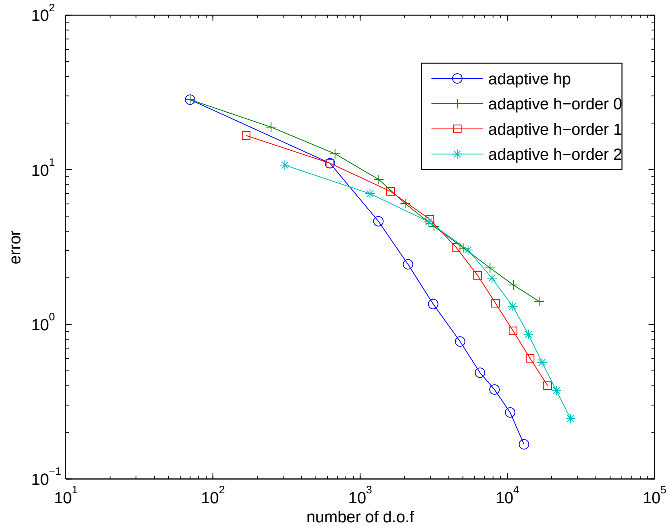


FIGURE 7. Convergence history for adaptive h- and hp-refinements on affine meshes.

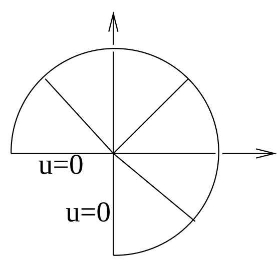
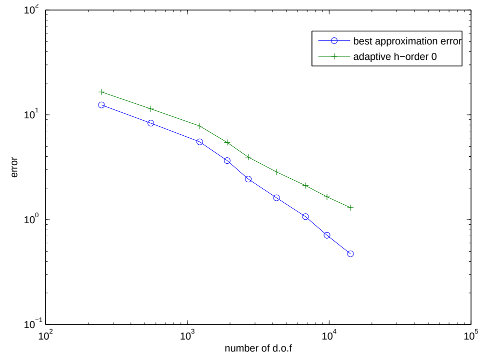


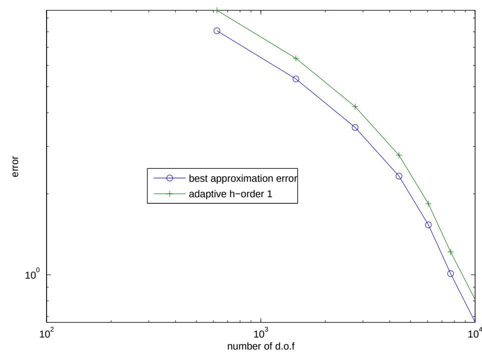
FIGURE 8. A circular L-shape domain with an initial mesh of six elements.

The analysis of curvilinear meshes for elasticity differs considerably from that for problems involving only grad-curl-div operators. Piola maps transform gradients, curls and divergence in the physical domain into the corresponding gradients, curls and divergence in the reference domain. Consequently, problems involving the grad, curl or div operators only (e.g. Maxwell equations or the mixed formulation for a scalar elliptic problem) can be reformulated in the parametric domain at the expense of introducing material anisotropies reflecting the geometric parametrizations. This is not the case for elasticity where the strain tensor (symmetric part of the displacement gradient) in the physical domain does not transform into the symmetric part of the displacement gradient in the reference domain[3]. Consequently, the analysis for affine meshes cannot be simply reproduced for curvilinear ones, and new interpolation operators have to be carefully drafted. We have managed to prove only an asymptotic stability for the curvilinear meshes.

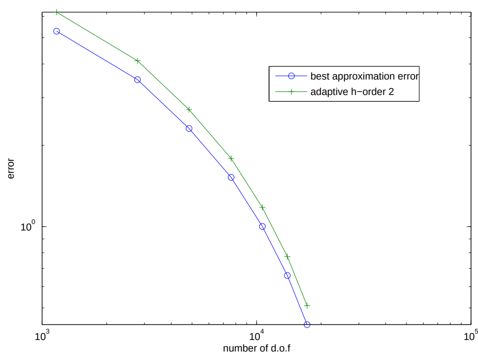
[3]The same problem is encountered in the case of complex stretchings introduced by Perfectly Matched Layers, see e.g. [13].



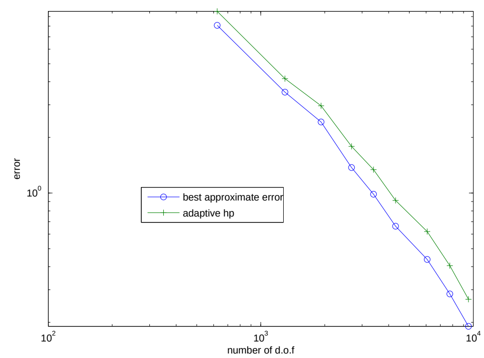
(a) Adaptive h-refinements, $p = 0$.



(b) Adaptive h-refinements, $p = 1$.



(c) Adaptive h-refinements, $p = 2$.



(d) Adaptive hp-refinements.

FIGURE 9. Comparison of FE error for adaptively refined curvilinear meshes with the best approximation error.

Presented numerical results on h-adaptivity go beyond our analysis as we use 1-irregular meshes with hanging nodes supported by an existing hp software.

Finally, the paper presents only a two-dimensional result. We continue working on the 3D case using different ideas and hope to present new results soon.

APPENDIX A. MESH GENERATION

We assume that the domain Ω is a (curvilinear) polygon, and that it can be meshed with a regular family $(\mathcal{T}_h)_h$ of C^0-compatible meshes (i.e. $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} T$, for all h) that satisfy the regularity assumptions discussed in the previous section.

We will outline now shortly how one can generate such meshes in practice. Suppose the domain Ω has been meshed with a C^0-compatible initial mesh $\mathcal{T}_{\text{int}}$, $\overline{\Omega} = \bigcup_{i=1}^m T_i$, where $\{T_i\}_{i=1}^m$ are curved triangles of class $C^{1,1}$. We denote by $\{G_1, \cdots, G_m\}$ the mappings from $\hat{T}$ to $\{T_1, \cdots, T_m\}$. Then $G_i \in C^{1,1}(\hat{T})$ and $G_i^{-1} \in C^1(T_i)$ for any $1 \le i \le m$. For examples of techniques to generate an initial mesh satisfying the assumptions above, see [12].

Lemma A.1. *Let $T = G_T(\hat{T})$ be a closed triangle in $\mathbb{R}^2$ with $G_T \in C^{1,1}(\hat{T})$. Let $\check{\sigma} > 0$ be a positive constant. For any $\check{h} > 0$, we denote by $\check{T}_{\check{h}}$ any triangle contained*

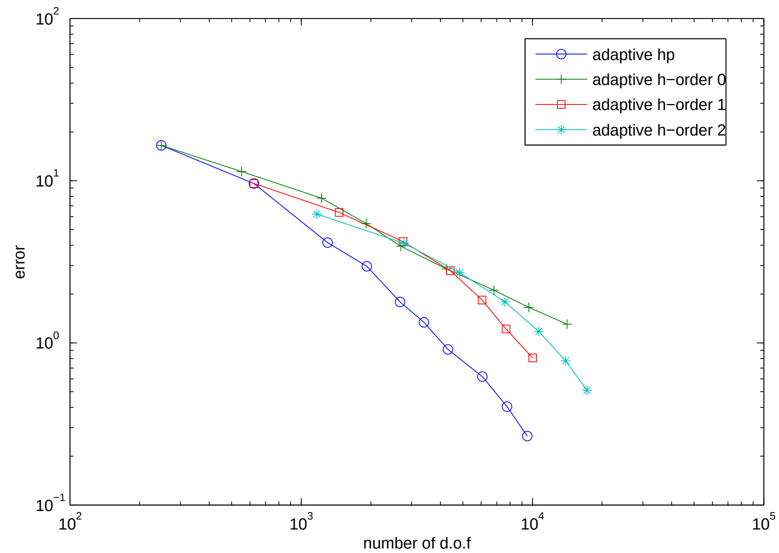


FIGURE 10. Convergence history for adaptive h- and hp-refinements on curvilinear meshes.

in $\hat{T}$ such that the diameter of $\check{T}_{\check{h}}$ is $\check{h}$, and $\check{h}/\check{\rho} \le \check{\sigma}$ where $\check{\rho}$ is the diameter of the sphere inscribed in $\check{T}_{\check{h}}$. Let $\hat{T} \ni \hat{\mathbf{x}} \to H\hat{\mathbf{x}} = \check{B}\hat{\mathbf{x}} + \check{b}$ be an affine mapping from $\hat{T}$ onto $\check{T}_{\check{h}}$. Let $\hat{\mathbf{p}}$ be the centroid of $\hat{T}$. We put $B = D(G_T \circ H)(\hat{\mathbf{p}})$, $b = (G_T \circ H)(\hat{\mathbf{p}})$, and $\Psi(\hat{\mathbf{x}}) = (G_T \circ H)(\hat{\mathbf{x}}) - \check{B}(\hat{\mathbf{x}} - \hat{\mathbf{p}}) - b$.

Then, we have

$$(\sup_{\hat{\mathbf{x}} \in \hat{T}} \|D\Psi(\hat{\mathbf{x}})\|)\|B^{-1}\| \le C\check{h},$$

where C is a constant independent of $\check{h}$, and $\check{T}_{\check{h}}$.

Proof. Obviously, $B = DG_T(H(\hat{\mathbf{p}}))\check{B}$. So $B^{-1} = \check{B}^{-1}(DG_T(H(\hat{\mathbf{p}})))^{-1}$. We have

$$D\Psi(\hat{\mathbf{x}}) = (DG_T(H(\hat{\mathbf{x}})) - DG_T(H(\hat{\mathbf{p}})))\check{B}.$$

Since G_T is a C^1-diffeomorphism from $\hat{T}$ onto T, we have $\sup_{\hat{\mathbf{x}} \in \hat{T}} \|(DG_T(\hat{\mathbf{x}}))^{-1}\| < \infty$. With $G_T \in C^{1,1}(\hat{T})$, we also have

$$\|DG_T(H(\hat{\mathbf{x}})) - DG_T(H(\hat{\mathbf{p}}))\| \le M\|H(\hat{\mathbf{x}}) - H(\hat{\mathbf{p}})\| \quad \text{for any } \hat{\mathbf{x}} \in \hat{T}$$

where M is the Lipschitz constant for all first order derivatives of G_T on $\hat{T}$.

With $\check{h}/\check{\rho} \le \check{\sigma}$, we obtain that $\|\check{B}\| \cdot \|\check{B}^{-1}\| \le \dfrac{c}{\check{\sigma}}$, for some $c > 0$. With $\check{h}$ denoting the diameter of $\check{T}_{\check{h}}$, we have $\|H(\hat{\mathbf{x}}) - H(\hat{\mathbf{p}})\| \le \check{h}$ for any $\hat{\mathbf{x}} \in \check{T}_{\check{h}}$. The definition of Ψ implies then

$$(\sup_{\hat{\mathbf{x}} \in \hat{T}} \|D\Psi(\hat{\mathbf{x}})\|)\|B^{-1}\| \le \frac{c}{\check{\sigma}} M(\sup_{\hat{\mathbf{y}} \in \hat{T}} \|(DG_T(\hat{\mathbf{y}}))^{-1}\|)\check{h}.$$

Setting $C = \dfrac{c}{\check{\sigma}} M(\sup_{\hat{\mathbf{y}} \in \hat{T}} \|(DG_T(\hat{\mathbf{y}}))^{-1}\|)$ finishes the proof. $\square$

Let $\hat{T}_i$ be now multiple copies of the reference triangle corresponding to the initial mesh, $T_i = G_i(\hat{T}_i)$, $i = 1, \ldots, m$.

Definition A.2. (Regular triangular meshes in the reference space) A family of triangulations $(\hat{\mathcal{T}}_{i,\check{h}})_{\check{h}}$ of reference triangles $\hat{T}_i$ is said to be regular provided two conditions are satisfied:

(i) Partitions of edges of $\hat{T}_i$ mapped into the same edge in the physical space are identical.

(ii) $\sup_{\check{h}} \sup_{\check{T} \in \hat{\mathcal{T}}_{i,\check{h}}} \check{h}/\check{\rho} < \infty$, where $\check{h}$ and $\check{\rho}$ are the outer and inner diameters of $\check{T}$.

Obviously, uniform refinements of reference triangles[4] are regular. A number of adaptive refinement algorithms produces regular meshes as well. To this class belong e.g. Rivara's algorithm (bisection by the longest edge), Arnold's algorithm (bisection by the newest edge), the Delaunay triangulation (see [8]).

Using Lemma A.1 and the fact that G_i is C^1-diffeomorphism from $\hat{T}$ to T_i, $1 \le i \le m$, we easily conclude that any regular refinements in the reference space produce curvilinear meshes that satisfy our mesh regularity assumptions.

Appendix B. Properties of Sobolev spaces on curved and reference triangles

Lemma B.1. *Let T be a curved triangle. For any $\omega \in H^1(T;\mathbb{R}^2)$, we define $\hat{\omega}(\hat{\mathbf{x}})$ on $\hat{T}$ by*

$$\omega(\mathbf{x}) = \frac{DG_T(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}}))}\hat{\omega}(\hat{\mathbf{x}}) \quad \hat{\mathbf{x}} \in \hat{T}.$$

Then $\hat{\omega}(\hat{\mathbf{x}}) \in H(\hat{T})$. Divergence transforms by the classical Piola's rule:

$$div\omega(\mathbf{x}) = (\det(DG_T(\hat{\mathbf{x}})))^{-1} div_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})$$

for $\hat{\mathbf{x}} \in \hat{T}$ almost everywhere.

Proof. Notice that $\hat{\omega}(\hat{\mathbf{x}}) = \det(DG_T(\hat{\mathbf{x}}))(DG_T(\hat{\mathbf{x}}))^{-1}\omega(\mathbf{x})$ for any $\hat{\mathbf{x}} \in \hat{T}$. It is straightforward to see that $\det(DG_T(\hat{\mathbf{x}}))(DG_T(\hat{\mathbf{x}}))^{-1}$ is a matrix whose entries contain first order partial derivatives of $G_T(\hat{x})$. Notice that

$$\hat{\omega}(\hat{\mathbf{x}}) = \det(DG_T(\hat{\mathbf{x}}))(DG_T(\hat{\mathbf{x}}))^{-1}\omega(\mathbf{x})$$

is the standard pull back mapping from $H(\mathrm{div}, T)$ to $H(\mathrm{div}_{\hat{\mathbf{x}}}, \hat{T})$. So we immediately have $\mathrm{div}\omega(\mathbf{x}) = \dfrac{1}{\det(DG_T(\hat{\mathbf{x}}))}\mathrm{div}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})$ for $\hat{\mathbf{x}} \in \hat{T}$ almost everywhere. Since G_T is a C^1-diffeomorphism from $\hat{T}$ to T, we can conclude that $\hat{\omega}(\hat{\mathbf{x}}) \in H(\mathrm{div}, \hat{T})$.

Let $e \in \triangle_1(T)$. We denote by $\zeta(s) : [0,1] \to \hat{e}$, the local affine parameterization of $\hat{e}$. We have then

$$\begin{aligned}
\|\hat{\omega}\|^2_{L^2(\hat{e})} &= \int_{[0,1]} (\hat{\omega}(\zeta(s)))^\top \hat{\omega}(\zeta(s)) \|\dot{\zeta}(s)\| ds \\
&= \int_{[0,1]} (\det(DG_T(\zeta(s))))^2 \omega(G_T(\zeta(s)))^\top [DG_T(\zeta(s))^{-\top} DG_T(\zeta(s))^{-1}] \\
&\qquad \omega(G_T(\zeta(s))) \|(DG_T(\zeta(s)))^{-1}[DG_T(\zeta(s))\dot{\zeta}(s)]\| ds.
\end{aligned}$$

[4]With the same number of divisions for each reference triangle.

Since G_T is a C^1-diffeomorphism from $\hat{T}$ to T, we can conclude that $\hat{\omega}|_{\partial\hat{T}} \in L^2(\partial\hat{T};\mathbb{R}^2)$. So $\hat{\omega} \in H(\hat{T})$. $\square$

Lemma B.2. *There exist $\delta > 0$ and $C > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$,*

$$\|\omega\|_{L^2(T)} \le C\|\hat{\omega}\|_{L^2(\hat{T})}, \forall \omega \in L^2(T;\mathbb{R}^2),$$

where $\dfrac{DG_T(\hat{\mathbf{x}})\hat{\omega}(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}}))} = \omega(\mathbf{x})$ *for any $\hat{\mathbf{x}} \in \hat{T}$. If T is a triangle for any $T \in \mathcal{T}_h$ and $h > 0$, then the above inequality holds for any $h > 0$.*

Proof.

$$\begin{aligned}\|\omega\|_{L^2(T)} &= \int_T \frac{1}{\det(DG_T(\hat{\mathbf{x}}))^2}[DG_T(\hat{\mathbf{x}})\hat{\omega}(\hat{\mathbf{x}})]^\top DG_T(\hat{\mathbf{x}})\hat{\omega}(\hat{\mathbf{x}})d\mathbf{x}\\ &= \int_{\hat{T}} \det((DG_T)^{-1})\hat{\omega}^\top[DG_T^\top DG_T]\hat{\omega}d\hat{\mathbf{x}}\\ &= \int_{\hat{T}} \det(DG_T^{-1}B_T)\det(B_T^{-1})\hat{\omega}^\top B_T^\top[B_T^{-\top}DG_T^\top DG_T B_T^{-1}]B_T\hat{\omega}d\hat{\mathbf{x}}.\end{aligned}$$

Since $c_h \to 0$ as $h \to 0$, $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}\|B_T^{-\top}DG_T^\top DG_T B_T^{-1} - I\| = 0$. By Lemma 7.9, we have $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}|\det(DG_T^{-1}B_T) - 1| = 0$.

Since $(\mathcal{T}_h)_h$ is regular, there is a constant $\sigma > 0$ such that $\sigma_1/\sigma_2 \le \sigma$ for any $T \in \mathcal{T}_h$, where σ_1 and σ_2 denote the biggest and smallest singular value of the corresponding matrix B_T. Then $\|B_T^\top\| = \|B_T\| = \sigma_1$ and $\det(B_T) = \sigma_1 \cdot \sigma_2$. So $\|B_T^\top\| \cdot \|B_T\|\det(B_T^{-1}) = \sigma_1/\sigma_2 \le \sigma$ for any $h > 0$ and any $T \in \mathcal{T}_h$.

We can conclude thus that there exist $\delta > 0$ and $C > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$,

$$\|\omega\|_{L^2(T)} \le C\|\hat{\omega}\|_{L^2(\hat{T})} \quad \forall \omega \in L^2(T;\mathbb{R}^2)$$

If, for all h, $T \in \mathcal{T}_h$ are (regular) triangles, the asymptotic argument is not necessary, and the above inequality holds for any $h > 0$. $\square$

Lemma B.3. *There exist $\delta > 0$ and $C > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$,*

$$\|\hat{\omega}\|^2_{H(div_{\hat{\mathbf{x}}},\hat{T})} + \|\hat{\omega}\|^2_{L^2(\partial\hat{T})} \le C\|\omega\|^2_{H^1(T)} \quad \forall \omega \in H^1(T;\mathbb{R}^2),$$

where $\dfrac{DG_T(\hat{\mathbf{x}})\hat{\omega}(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}}))} = \omega(\mathbf{x})$ *for any $\hat{\mathbf{x}} \in \hat{T}$. If T is a triangle for any $T \in \mathcal{T}_h$ and $h > 0$, then the above inequality holds for any $h > 0$.*

Proof.

$$\begin{aligned}\|\hat{\omega}\|^2_{L^2(\hat{T})} &= \int_{\hat{T}}(\det(DG_T(\hat{\mathbf{x}})))^2\omega(\mathbf{x})^\top[DG_T(\hat{\mathbf{x}})^{-T}DG_T(\hat{\mathbf{x}})^{-1}]\omega(\mathbf{x})d\hat{\mathbf{x}}\\ &= \int_T \det(DG_T(\hat{\mathbf{x}}))\omega(\mathbf{x})^\top[DG_T(\hat{\mathbf{x}})^{-T}DG_T(\hat{\mathbf{x}})^{-1}]\omega(\mathbf{x})d\mathbf{x}\\ &= \int_T \det(DG_T(\hat{\mathbf{x}})B_T^{-1})\det(B_T)\omega(\mathbf{x})^\top B_T^{-\top}\\ &\qquad [B_T^\top DG_T(\hat{\mathbf{x}})^{-T}DG_T(\hat{\mathbf{x}})^{-1}B_T]B_T^{-1}\omega(\mathbf{x})d\mathbf{x}.\end{aligned}$$

According to Lemma 7.8, $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}\|B_T^\top DG_T(\hat{\mathbf{x}})^{-T}DG_T(\hat{\mathbf{x}})^{-1}B_T - I\| = 0$. By Lemma 7.9, we have $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}|\det(DG_T(\hat{\mathbf{x}})B_T^{-1}) - 1| = 0$.

Since $(\mathcal{T}_h)_h$ is regular, there exists a constant $\sigma > 0$ such that $\sigma_1/\sigma_2 \le \sigma$ for any $T \in \mathcal{T}_h$, where σ_1 and σ_2 denote the biggest and smallest singular value of the matrix B_T. Then $\|B_T^{-\top}\| = \|B_T^{-1}\| = \sigma_2^{-1}$ and $\det(B_T) = \sigma_1 \cdot \sigma_2$. So $\|B_T^{-\top}\| \cdot \|B_T^{-1}\| \det(B_T) = \sigma_1/\sigma_2 \le \sigma$. Consequently, there exist $\delta_1 > 0$ and $C_1 > 0$ such that, for any $h < \delta_1$ and $T \in \mathcal{T}_h$,

$$\begin{aligned}&\det(DG_T(\hat{\mathbf{x}})B_T^{-1})\det(B_T)\omega(\mathbf{x})^\top B_T^{-\top}[B_T^\top DG_T(\hat{\mathbf{x}})^{-T} DG_T(\hat{\mathbf{x}})^{-1}B_T]B_T^{-1}\omega(\mathbf{x})\\ &\le C_1^2 \omega(\mathbf{x})^\top \omega(\mathbf{x})\end{aligned}$$

for all $\omega \in H^1(T;\mathbb{R}^2), \mathbf{x} \in T$. We can conclude that, for any $h < \delta_1$ and $T \in \mathcal{T}_h$,

$$\|\hat{\omega}\|_{L^2(\hat{T})} \le C_1 \|\omega\|_{L^2(T)} \quad \forall \omega \in H^1(T;\mathbb{R}^2)$$

According to Lemma B.1, we have

$$\begin{aligned}\|\mathrm{div}_{\hat{\mathbf{x}}}\hat{\omega}\|^2_{L^2(\hat{T})} &= \int_{\hat{T}} (\det(DG_T(\hat{\mathbf{x}})))^2(\mathrm{div}\omega(\mathbf{x}))^2 d\hat{\mathbf{x}} = \int_T \det(DG_T(\hat{\mathbf{x}}))(\mathrm{div}\omega(\mathbf{x}))^2 d\mathbf{x}\\ &= \int_T \det(DG_T(\hat{\mathbf{x}})B_T^{-1})\det(B_T)(\mathrm{div}\omega(\mathbf{x}))^2 d\mathbf{x}.\end{aligned}$$

By Lemma 7.9, $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}} |\det(DG_T(\hat{\mathbf{x}})B_T^{-1}) - 1| = 0$. Obviously, $\det(B_T) \le \tilde{h}_T^2$. There must exist then $\delta_2 > 0$ and $C_2 > 0$ such that, for any $h < \delta_2$ and $T \in \mathcal{T}_h$,

$$\det(DG_T(\hat{\mathbf{x}})B_T^{-1})\det(B_T)(\mathrm{div}\omega(\mathbf{x}))^2 \le C_2^2 \tilde{h}_T^2 (\mathrm{div}\omega(\mathbf{x}))^2 \quad \forall \omega \in H^1(T;\mathbb{R}^2), \mathbf{x} \in T.$$

We conclude that, for any $h < \delta_2$ and $T \in \mathcal{T}_h$,

$$\|\mathrm{div}_{\hat{\mathbf{x}}}\hat{\omega}\|_{L^2(\hat{T})} \le C_2 \tilde{h}_T \|\mathrm{div}\omega\|_{L^2(T)} \quad \forall \omega \in H^1(T;\mathbb{R}^2).$$

We take now an arbitrary $e \in \triangle_1(T)$. We denote by $\zeta(s) : [0,1] \to \hat{e}$ the local affine parameterization of $\hat{e}$. We have then

$$\begin{aligned}\|\hat{\omega}\|^2_{L^2(\hat{e})} &= \int_{[0,1]} (\hat{\omega}(\zeta(s)))^\top \hat{\omega}(\zeta(s)) \|\dot{\zeta}(s)\| ds\\ &= \int_{[0,1]} (\det(DG_T(\zeta(s))))^2 \omega(G_T(\zeta(s)))^\top [DG_T(\zeta(s))^{-\top} DG_T(\zeta(s))^{-1}]\\ &\qquad \omega(G_T(\zeta(s)))\|\dot{\zeta}(s)\| ds\\ &= \int_{[0,1]} (\det(DG_T(\zeta(s)))B_T^{-1})^2 (\det(B_T))^2 \omega(G_T(\zeta(s)))^\top B_T^{-\top}\\ &\qquad [B_T^\top DG_T(\zeta(s))^{-\top} DG_T(\zeta(s))^{-1} B_T] B_T^{-1}\omega(G_T(\zeta(s)))\|\dot{\zeta}(s)\| ds.\end{aligned}$$

According to Lemma 7.8, we have that

$$\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{e\in T} \sup_{s\in[0,1]} \|B_T^\top DG_T(\zeta(s))^{-\top} DG_T(\zeta(s))^{-1} B_T - I\| = 0.$$

By Lemma 7.9, $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{e\in T}\sup_{s\in[0,1]} |(\det(DG_T(\zeta(s)))B_T^{-1})^2 - 1| = 0$.

Consider again the singular values of B_T, $\sigma_1 \geq \sigma_2$. Then $\|B_T^{-\top}\| = \|B_T^{-1}\| = \sigma_2^{-1}$, and $\det(B_T) = \sigma_1 \cdot \sigma_2$. So $\|B_T^{-\top}\| \cdot \|B_T^{-1}\|(\det(B_T))^2 = (\sigma_1)^2 \leq \tilde{h}_T^2$. Consequently, there exist $\delta_3 > 0$ and $C_3 > 0$ such that, for any $h < \delta_3$ and $T \in \mathcal{T}_h$,

$$\begin{aligned}
&(\det(DG_T(\zeta(s)))B_T^{-1})^2(\det(B_T))^2\omega(G_T(\zeta(s)))^\top B_T^{-\top} \\
&[B_T^\top DG_T(\zeta(s))^{-\top} DG_T(\zeta(s))^{-1}B_T]B_T^{-1}\omega(G_T(\zeta(s))) \\
\leq &C_3\tilde{h}_T^2\omega(\mathbf{x})^\top\omega(\mathbf{x}) \quad \forall\omega \in H^1(T;\mathbb{R}^2), e \in \triangle_1(T), s \in [0,1]
\end{aligned}$$

and

$$\begin{aligned}
\|\dot{\zeta}(s)\| =&\|B_T^{-1}[B_T DG_T(\zeta(s))^{-1}] \cdot [DG_T(\zeta(s))\dot{\zeta}(s)]\| \\
\leq &C_3\tilde{h}_T^{-1}\|DG_T(\zeta(s))\dot{\zeta}(s)\| \quad \forall e \in \triangle_1(T), s \in [0,1].
\end{aligned}$$

We can conclude that, for any $h < \delta_3$ and $T \in \mathcal{T}_h$,

$$\begin{aligned}
\|\hat{\omega}\|_{L^2(\hat{e})}^2 &\leq C_3^2\tilde{h}_T\int_{[0,1]}\omega(G_T(\zeta(s)))^\top\omega(G_T(\zeta(s)))\|DG_T(\zeta(s))\dot{\zeta}(s)\|ds \\
&= C_3^2\tilde{h}_T\|\omega\|_{L^2(e)}^2.
\end{aligned}$$

Obviously, $\sup_h \sup_{T\in\mathcal{T}_h} \tilde{h}_T < \infty$. Since $\omega \in H^1(T;\mathbb{R}^2)$, we can use the Trace Theorem to conclude that there exist $\delta > 0$ and $C > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$,

$$\|\hat{\omega}\|_{H(\operatorname{div}_{\hat{\mathbf{x}}},\hat{T})}^2 + \|\hat{\omega}\|_{L^2(\partial\hat{T})}^2 \leq C\|\omega\|_{H^1(T)}^2 \quad \forall\omega \in H^1(T;\mathbb{R}^2).$$

It is easy to see that, if T is a (regular) triangle for any $T \in \mathcal{T}_h$ and $h > 0$, then the inequality above holds for any $h > 0$. $\square$

Lemma B.4. *There exist $\delta > 0$ and $C > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$,*

$$\|\mathit{curl}\omega\|_{L^2(T)}^2 \leq C\tilde{h}_T^{-2}\|\mathit{curl}_{\hat{\mathbf{x}}}\hat{\omega}\|_{L^2(\hat{T})}^2 \quad \forall\omega \in H^1(T;\mathbb{R}^2)$$

where $\dfrac{B_T\hat{\omega}(\hat{\mathbf{x}})}{\det(B_T)} = \omega(\mathbf{x})$ *for any* $\hat{\mathbf{x}} \in \hat{T}$. *If T is a triangle for any $T \in \mathcal{T}_h$ and $h > 0$, then the above inequality holds for any $h > 0$.*

Proof. We have,

$$\begin{aligned}
\operatorname{curl}\omega(\mathbf{x}) =&\frac{B_T}{\det(B_T)}(\operatorname{curl}\hat{\omega})(\hat{\mathbf{x}}) = \frac{1}{\det(B_T DG_T(\hat{\mathbf{x}}))}B_T\operatorname{curl}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})(DG_T(\hat{\mathbf{x}}))^\top \\
=&\frac{(\det(B_T^{-1}))^2}{\det(B_T^{-1}DG_T(\hat{\mathbf{x}}))}B_T\operatorname{curl}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})B_T^\top(DG_T(\hat{\mathbf{x}})B_T^{-1})^\top.
\end{aligned}$$

and

$$\begin{aligned}
\|\operatorname{curl}\omega\|_{L^2(T)}^2 =&\int_T \frac{(\det(B_T^{-1}))^4}{(\det(B_T^{-1}DG_T(\hat{\mathbf{x}})))^2}(DG_T(\hat{\mathbf{x}})B_T^{-1})B_T(\operatorname{curl}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}}))^\top \\
&[B_T^\top B_T]\operatorname{curl}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})B_T^\top(DG_T(\hat{\mathbf{x}})B_T^{-1})^\top d\mathbf{x} \\
=&\int_{\hat{T}} \frac{(\det(B_T^{-1}))^3}{(\det(B_T^{-1}DG_T(\hat{\mathbf{x}})))^2}(DG_T(\hat{\mathbf{x}})B_T^{-1})B_T(\operatorname{curl}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}}))^\top \\
&[B_T^\top B_T]\operatorname{curl}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})B_T^\top(DG_T(\hat{\mathbf{x}})B_T^{-1})^\top d\hat{\mathbf{x}}.
\end{aligned}$$

Since $c_h \to 0$ as $h \to 0$, $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}\|DG_T(\hat{\mathbf{x}})B_T^{-1} - I\| = 0$.
By Lemma 7.9, $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}|(\det(B_T^{-1}DG_T(\hat{\mathbf{x}})))^2 - 1| = 0$.

Since $(\mathcal{T}_h)_h$ is regular, there exists a constant $\sigma > 0$ such that $\sigma_1/\sigma_2 \leq \sigma$ for any $T \in \mathcal{T}_h$, with $\sigma_1 \geq \sigma_2$ denoting the singular values of matrix B_T. Then $\|B_T^\top\| = \|B_T\| = \sigma_1$ and $\det(B_T^{-1}) = \sigma_1^{-1} \cdot \sigma_2^{-1}$. So $\|B_T^\top\|^2 \cdot \|B_T\|^2 (\det(B_T^{-1}))^3 = \sigma_1/\sigma_2^3 \leq \sigma/\sigma_2^2 \leq c'\tilde{h}_T^{-2}$ for some constant $c' > 0$.

Consequently, there exist $\delta_1 > 0$ and $C > 0$ such that, for any $h < \delta_1$ and $T \in \mathcal{T}_h$,

$$\begin{aligned}
&\int_{\hat{T}} \frac{(\det(B_T^{-1}))^3}{(\det(B_T^{-1} DG_T(\hat{\mathbf{x}})))^2} (DG_T(\hat{\mathbf{x}}) B_T^{-1}) B_T (\mathrm{curl}_{\hat{\mathbf{x}}} \hat{\omega}(\hat{\mathbf{x}}))^\top \\
&\qquad [B_T^\top B_T] \mathrm{curl}_{\hat{\mathbf{x}}} \hat{\omega}(\hat{\mathbf{x}}) B_T^\top (DG_T(\hat{\mathbf{x}}) B_T^{-1})^\top d\hat{\mathbf{x}} \\
\leq & C\tilde{h}_T^{-2} \int_{\hat{T}} \|(\mathrm{curl}_{\hat{\mathbf{x}}} \hat{\omega}(\hat{\mathbf{x}}))^\top (\mathrm{curl}_{\hat{\mathbf{x}}} \hat{\omega}(\hat{\mathbf{x}}))\| d\hat{\mathbf{x}}.
\end{aligned}$$

Again, it is easy to see that, if T is a (regular) triangle for any $T \in \mathcal{T}_h$ and $h > 0$, then the above inequality holds for any $h > 0$. This finishes the proof. $\square$

Lemma B.5. *There exist $\delta > 0$ and $C > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$,*

$$\|\hat{\omega}\|^2_{H(curl_{\hat{\mathbf{x}}}, \hat{T})} \leq C(\|\omega\|^2_{L^2(T)} + \tilde{h}_T^2 \|curl \omega\|^2_{L^2(T)}) \quad \forall \omega \in H^1(T; \mathbb{R}^2)$$

where $\dfrac{B_T \hat{\omega}(\hat{\mathbf{x}})}{\det(B_T)} = \omega(\mathbf{x})$ for any $\hat{\mathbf{x}} \in \hat{T}$. If T is a triangle for any $T \in \mathcal{T}_h$ and $h > 0$, then the above inequality holds for any $h > 0$.

Proof. We have,

$$\|\hat{\omega}\|^2_{H(\mathrm{curl}_{\hat{\mathbf{x}}}, \hat{T})} = \|\hat{\omega}\|^2_{L^2(\hat{T})} + \|\mathrm{curl}_{\hat{\mathbf{x}}} \hat{\omega}\|^2_{L^2(\hat{T})},$$

and

$$\begin{aligned}
\|\hat{\omega}\|^2_{L^2(\hat{T})} &= \int_{\hat{T}} (\det(B_T))^2 \omega(\mathbf{x})^\top B_T^{-\top} B_T^{-1} \omega(\mathbf{x}) d\hat{\mathbf{x}} \\
&= \int_T \det(B_T) \det(B_T DG_T(\hat{\mathbf{x}})^{-1}) \omega(\mathbf{x})^\top B_T^{-\top} B_T^{-1} \omega(\mathbf{x}) d\mathbf{x}.
\end{aligned}$$

By Lemma 7.9, $\lim_{h \to 0} \sup_{T \in \mathcal{T}_h} \sup_{\hat{\mathbf{x}} \in \hat{T}} |\det(B_T DG_T(\hat{\mathbf{x}})^{-1}) - 1| = 0$.

Since $(\mathcal{T}_h)_h$ is regular, there exists a constant $\sigma > 0$ such that $\sigma_1/\sigma_2 \leq \sigma$ for any $T \in \mathcal{T}_h$, where $\sigma_1 \geq \sigma_2$ are the singular values of matrix B_T. Then $\|B_T^{-\top}\| = \|B_T^{-1}\| = \sigma_2^{-1}$ and $\det(B_T) = \sigma_1 \cdot \sigma_2$. So, $\|B_T^{-\top}\| \cdot \|B_T^{-1}\| \det(B_T) = \sigma_1/\sigma_2 \leq \sigma$. Consequently, there exist $\delta_1 > 0$ and $C_1 > 0$ such that, for any $h < \delta_1$ and $T \in \mathcal{T}_h$,

$$\det(B_T) \det(B_T DG_T(\hat{\mathbf{x}})^{-1}) \omega(\mathbf{x})^\top B_T^{-\top} B_T^{-1} \omega(\mathbf{x}) \leq C_1^2 \omega(\mathbf{x})^\top \omega(\mathbf{x})$$

for any $\omega \in H^1(T; \mathbb{R}^2), \mathbf{x} \in T$. We can conclude that, for any $h < \delta_1$ and $T \in \mathcal{T}_h$,

$$\|\hat{\omega}\|_{L^2(\hat{T})} \leq C_1 \|\omega\|_{L^2(T)}, \forall \omega \in H^1(T; \mathbb{R}^2).$$

At the same time,

$$\begin{aligned}
\mathrm{curl}_{\hat{\mathbf{x}}} \hat{\omega}(\hat{\mathbf{x}}) &= \det(B_T) B_T^{-1} \mathrm{curl}_{\hat{\mathbf{x}}} \omega(\mathbf{x}) \\
&= \det(B_T) B_T^{-1} \mathrm{curl}_{\mathbf{x}} \omega(\mathbf{x}) (DG_T(\hat{\mathbf{x}}))^{-\top} \det(DG_T(\hat{\mathbf{x}})) \\
&= \det(B_T) B_T^{-1} \mathrm{curl}_{\mathbf{x}} \omega(\mathbf{x}) B_T^{-\top} (B_T^\top DG_T(\hat{\mathbf{x}})^{-\top}) \det(DG_T(\hat{\mathbf{x}})) \\
&= \det(B_T) B_T^{-1} \mathrm{curl}_{\mathbf{x}} \omega(\mathbf{x}) B_T^{-\top} (DG_T(\hat{\mathbf{x}})^{-1} B_T)^\top \det(DG_T(\hat{\mathbf{x}})).
\end{aligned}$$

and

$$\begin{aligned}\|\mathrm{curl}_{\hat{\mathbf{x}}}\hat{\omega}\|^2_{L^2(\hat{T})} &= \int_{\hat{T}} \det(B_T)^2 \det(DG_T(\hat{\mathbf{x}}))^2 (DG_T(\hat{\mathbf{x}})^{-1}B_T)B_T^{-1}(\mathrm{curl}_{\mathbf{x}}\omega(\mathbf{x}))^\top \\ &\qquad B_T^{-\top}B_T^{-1}\mathrm{curl}_{\mathbf{x}}\omega(\mathbf{x})B_T^{-\top}(DG_T(\hat{\mathbf{x}})^{-1}B_T)^\top d\hat{\mathbf{x}} \\ &= \int_{\hat{T}} \det(B_T)^2 \det(DG_T(\hat{\mathbf{x}}))(DG_T(\hat{\mathbf{x}})^{-1}B_T)B_T^{-1}(\mathrm{curl}_{\mathbf{x}}\omega(\mathbf{x}))^\top \\ &\qquad B_T^{-\top}B_T^{-1}\mathrm{curl}_{\mathbf{x}}\omega(\mathbf{x})B_T^{-\top}(DG_T(\hat{\mathbf{x}})^{-1}B_T)^\top d\mathbf{x} \\ &= \int_{\hat{T}} \det(B_T)^3 \det(B_T^{-1}DG_T(\hat{\mathbf{x}}))(DG_T(\hat{\mathbf{x}})^{-1}B_T)B_T^{-1}(\mathrm{curl}_{\mathbf{x}}\omega(\mathbf{x}))^\top \\ &\qquad B_T^{-\top}B_T^{-1}\mathrm{curl}_{\mathbf{x}}\omega(\mathbf{x})B_T^{-\top}(DG_T(\hat{\mathbf{x}})^{-1}B_T)^\top d\mathbf{x}.\end{aligned}$$

According to Lemma 7.8, $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}\|DG_T(\hat{\mathbf{x}})^{-1}B_T - I\| = 0$.
By Lemma 7.9, $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h}\sup_{\hat{\mathbf{x}}\in\hat{T}}|\det(B_T^{-1}DG_T(\hat{\mathbf{x}})) - 1| = 0$.
Since $(\mathcal{T}_h)_h$ is regular, $\det(B_T)^3\|B_T^{-\top}\|^2\cdot\|B_T^{-1}\|^2 \le \sigma\tilde{h}_T$, for some constant $\sigma > 0$.

There exist thus $\delta_2 > 0$ and $C_2 > 0$ such that, for any $h < \delta_2$ and $T \in \mathcal{T}_h$,

$$\begin{aligned}&\det(B_T)^3 \det(B_T^{-1}DG_T(\hat{\mathbf{x}}))(DG_T(\hat{\mathbf{x}})^{-1}B_T)B_T^{-1}(\mathrm{curl}_{\mathbf{x}}\omega(\mathbf{x}))^\top \\ &\quad B_T^{-\top}B_T^{-1}\mathrm{curl}_{\mathbf{x}}\omega(\mathbf{x})B_T^{-\top}(DG_T(\hat{\mathbf{x}})^{-1}B_T)^\top d\mathbf{x} \\ \le & C_2^2\tilde{h}_T^2\|(\mathrm{curl}_{\mathbf{x}}\omega(\mathbf{x}))^\top\mathrm{curl}_{\mathbf{x}}\omega(\mathbf{x})\| \quad \forall\omega\in H^1(T;\mathbb{R}^2), \mathbf{x}\in T.\end{aligned}$$

We conclude that, for any $h < \delta_2$ and $T \in \mathcal{T}_h$,

$$\|\mathrm{curl}_{\hat{\mathbf{x}}}\hat{\omega}\|_{L^2(\hat{T})} \le C_2\tilde{h}_T\|\mathrm{curl}\omega\|_{L^2(T)}, \forall\omega\in H^1(T;\mathbb{R}^2).$$

Again, it is easy to see that, if T is a triangle for any $T \in \mathcal{T}_h$ and $h > 0$, then the inequality above holds for any $h > 0$. This ends the proof. □

Lemma B.6. *Let T be a curved triangle. Let $u(\mathbf{x})$ be defined on T and $\hat{u}(\hat{\mathbf{x}})$ be defined on $\hat{T}$ and*

$$u = \hat{u}\circ G_T^{-1} \quad \text{or} \quad \hat{u} = u\circ G_T$$

where G_T is the element map. Then $u(\mathbf{x}) \in L^2(T)$ if and only if $\hat{u}(\hat{x}) \in L^2(\hat{T})$.

Proof. This is an immediate consequence of the fact that G_T is a C^1-diffeomorphism. □

Appendix C. Proofs of results from Section 8

Proof of Lemma 9.5

Proof. We put $r = \tilde{r}(T)$. We assume $\{\hat{\xi}_1(\hat{\mathbf{x}}), \cdots, \hat{\xi}(\hat{\mathbf{x}})_{l_r}\}$ is a basis for $\mathcal{P}_{\tilde{r}}\Lambda^2(\hat{T})$. Then

$$\Pi^2_{\tilde{r},T}u(\mathbf{x}(\hat{\mathbf{x}})) = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))}\sum_{i=1}^{l_r}\alpha_i\hat{\xi}_i(\hat{\mathbf{x}})$$

and

$$P^2_{\tilde{r},T}u(\mathbf{x}(\hat{\mathbf{x}})) = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))}\sum_{i=1}^{l_r}\beta_i\hat{\xi}_i(\hat{\mathbf{x}})$$

Coefficients $(\alpha_1,\cdots,\alpha_{l_r})^\top$ and $(\beta_1,\cdots,\beta_{l_r})^\top$ are obtained by solving the following two linear systems,

$$A_1(\alpha_1,\cdots,\alpha_{l_r})^\top = \mathbf{b}_1 \quad \text{and} \quad A_2(\beta_1,\cdots,\beta_{l_r})^\top = \mathbf{b}_2$$

with

$$(A_1)_{ij} = \int_T \frac{\hat{\xi}_i(\hat{\mathbf{x}}(\mathbf{x}))\hat{\xi}_j(\hat{\mathbf{x}}(\mathbf{x}))d\mathbf{x}}{\det(DG_T(\hat{\mathbf{x}}(\mathbf{x})))} \quad (\mathbf{b}_1)_j = \int_T u(\mathbf{x})\hat{\xi}_j(\hat{\mathbf{x}}(\mathbf{x}))d\mathbf{x}$$

$$(A_2)_{ij} = \int_T \frac{\hat{\xi}_i(\hat{\mathbf{x}}(\mathbf{x}))\hat{\xi}_j(\hat{\mathbf{x}}(\mathbf{x}))d\mathbf{x}}{(\det(DG_T(\hat{\mathbf{x}}(\mathbf{x}))))^2} \quad (\mathbf{b}_2)_j = \int_T \frac{\hat{\xi}_j(\hat{\mathbf{x}}(\mathbf{x}))d\mathbf{x}}{\det(DG_T(\hat{\mathbf{x}}(\mathbf{x})))}u(\mathbf{x})$$

for $1 \le i, j \le l_r$. By pulling back to $\hat{T}$ we obtain, for any $1 \le i, j \le l_r$,

$$(A_1)_{ij} = \int_{\hat{T}} \hat{\xi}_i(\hat{\mathbf{x}})\hat{\xi}_j(\hat{\mathbf{x}})d\hat{\mathbf{x}} \quad (\mathbf{b}_1)_j = \det(B_T)\int_{\hat{T}} \det(B_T^{-1}DG_T(\hat{\mathbf{x}}))u(\mathbf{x})\hat{\xi}_j(\hat{\mathbf{x}}(\mathbf{x}))d\hat{\mathbf{x}}$$

$$(A_2)_{ij} = \int_{\hat{T}} \frac{\hat{\xi}_i(\hat{\mathbf{x}})\hat{\xi}_j(\hat{\mathbf{x}})d\hat{\mathbf{x}}}{\det(DG_T(\hat{\mathbf{x}}))} \quad (\mathbf{b}_2)_j = \int_{\hat{T}} u(\mathbf{x}(\mathbf{x}))\hat{\xi}_j(\hat{\mathbf{x}})d\hat{\mathbf{x}}$$

Since $\det(B_T)$ is a non-zero constant, $(\det(B_T)A_2)(\beta_1, \cdots, \beta_{l_r})^\top = \det(B_T)\mathbf{b}_2$. So we can redefine A_2 and $\mathbf{b}_2$ in the following way.

$$(A_2)_{ij} = \int_{\hat{T}} \frac{\det(B_T)}{\det(DG_T(\hat{\mathbf{x}}))}\hat{\xi}_i(\hat{\mathbf{x}})\hat{\xi}_j(\hat{\mathbf{x}})d\hat{\mathbf{x}} \quad (\mathbf{b}_2)_j = \det(B_T)\int_{\hat{T}} u(\mathbf{x}(\hat{\mathbf{x}}))\hat{\xi}_j(\hat{\mathbf{x}})d\hat{\mathbf{x}}$$

According to Lemma 7.9, $\lim_{h\to 0}\sup_{T\in\mathcal{T}_h} \|A_1 - A_2\| = 0$. And, for any $\varepsilon > 0$,

$$\begin{aligned}\lim_{h\to 0}\sup_{T\in\mathcal{T}_h} \|\mathbf{b}_1 - \mathbf{b}_2\|^2 \le &(\varepsilon\det(B_T))^2\int_{\hat{T}} u^2(\mathbf{x}(\hat{\mathbf{x}}))d\hat{\mathbf{x}} \\ =&\varepsilon^2\det(B_T)\int_T \det(B_T DG_T(\hat{\mathbf{x}}(\mathbf{x}))^{-1})u^2(\mathbf{x})d\mathbf{x} \\ \le&4\varepsilon^2\det(B_T)\|u\|^2_{L^2(T)}.\end{aligned}$$

The last inequality holds when h is small enough. This implies that

$$\lim_{h\to 0}\sup_{T\in\mathcal{T}_h} \|\mathbf{b}_1 - \mathbf{b}_2\| \le 2\varepsilon\sqrt{\det(B_T)}\|u\|_{L^2(T)}.$$

So, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $h \le \delta$ and $T \in \mathcal{T}_h$, we have

$$\|(\alpha_1 - \beta_1, \cdots, \alpha_{l_r} - \beta_{l_r})\| \le 3\varepsilon\sqrt{\det(B_T)}\|u\|_{L^2(T)}.$$

We have then

$$\begin{aligned}\|\Pi^2_{\tilde{r},T}u - P_{\tilde{r},T}u\|^2_{L^2(T)} =& \int_T \frac{1}{(\det(DG_T(\hat{\mathbf{x}}(\mathbf{x}))))^2}\sum_{i=1}^{l_r}\hat{\xi}_i^2(\hat{\mathbf{x}}(\mathbf{x}))(\alpha_i - \beta_i)^2 d\mathbf{x} \\ =& \int_{\hat{T}} \frac{1}{\det(DG_T(\hat{\mathbf{x}}))}\sum_{i=1}^{l_r}\hat{\xi}_i^2(\hat{\mathbf{x}})(\alpha_i - \beta_i)^2 d\hat{\mathbf{x}} \\ \le& c\varepsilon^2\|u\|^2_{L^2(T)}\int_{\hat{T}} \det(B_T DG_T(\hat{\mathbf{x}})^{-1})\sum_{i=1}^{l_r}\hat{\xi}_i^2(\hat{\mathbf{x}})d\hat{\mathbf{x}}\end{aligned}$$

Here c is a positive constant which depends on l_r only. By Lemma 7.9, there exists $M > 0$ such that, for any h small enough and any $T \in \mathcal{T}_h$,

$$\int_{\hat{T}} \det(B_T DG_T(\hat{\mathbf{x}})^{-1})\sum_{i=1}^{l_r}\hat{\xi}_i^2(\hat{\mathbf{x}})d\hat{\mathbf{x}} \le M^2$$

We can conclude that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$,

$$\|\Pi^2_{\tilde{r},T}u - P_{\tilde{r},T}u\|_{L^2(T)} \le \varepsilon\|u\|_{L^2(T)}, \forall u \in L^2(T)$$

Here $P_{\tilde{r},T}$ is the standard L^2-projection onto $\mathcal{P}_{\tilde{r}}\Lambda^2(T)$. $\square$

Proof of Lemma 9.9

Proof. Obviously, $\hat{\omega}(\hat{\mathbf{x}}) = \det(DG_T(\hat{\mathbf{x}}))(DG_T(\hat{\mathbf{x}}))^{-1}\omega(G_T(\hat{\mathbf{x}}))$. Using Lemma B.1, we can conclude that $\hat{\omega}(\hat{\mathbf{x}}) \in H(\hat{T})$.

By pulling back to $\hat{T}$ and using the definition of $\mathcal{P}_{\tilde{r}+1}\Lambda^1(T)$, we can see that (9.4) is the same as (9.7), and (9.3) is the same as (9.6). Thus, we only need to show that (9.5) is the same as (9.8). Since $\mathcal{T}_h$ is C^0-compatible, then $G_T(\zeta(s)) = \mathbf{x}_e(s)$ for any $s \in [0,1]$ where $\zeta : [0,1] \to \hat{e}$ is an affine local parameterization of $\hat{e}$.
We have then

$$
\begin{aligned}
&\int_{[0,1]} [\omega(\mathbf{x}_e(s)) \cdot \mathbf{n}(\mathbf{x}_e(s))]\hat{\eta}(s)\|\dot{\mathbf{x}}_e(s)\|ds \\
=&\int_{[0,1]} [\omega(G_T(\zeta(s))) \cdot \mathbf{n}(G_T(\zeta(s)))]\hat{\eta}(s)\|DG_T(\zeta(s))\dot{\zeta}(s)\|ds \\
=&\int_{[0,1]} \left[\frac{DG_T(\zeta(s))\hat{\omega}(\zeta(s))}{\det(DG_T(\zeta(s)))} \cdot \frac{(DG_T(\zeta(s)))^{-\top}\hat{\mathbf{n}}(\zeta(s))}{\|(DG_T(\zeta(s)))^{-\top}\hat{\mathbf{n}}(\zeta(s))\|}\right]\hat{\eta}(s)\|DG_T(\zeta(s))\dot{\zeta}(s)\|ds \\
=&\int_{[0,1]} \left[\frac{\hat{\omega}(\zeta(s))}{\det(DG_T(\zeta(s)))} \cdot \frac{\hat{\mathbf{n}}(\zeta(s))}{\|(DG_T(\zeta(s)))^{-\top}\hat{\mathbf{n}}(\zeta(s))\|}\right]\hat{\eta}(s)\|DG_T(\zeta(s))\dot{\zeta}(s)\|ds
\end{aligned}
$$

Notice that $\zeta(s) = c\hat{t}$ where $\hat{t}$ is a unit tangent vector along $\hat{e}$, and c is a nonzero constant, $\hat{\mathbf{n}}(\zeta(s)) = (\hat{t}_2, -\hat{t}_1)^\top$, and $(DG_T(\zeta(s)))^{-\top} = \dfrac{A}{\det(DG_T(\zeta(s)))}$ with

$$
A = \begin{bmatrix} (DG_T)_{22}(\zeta(s)) & -(DG_T)_{21}(\zeta(s)) \\ -(DG_T)_{12}(\zeta(s)) & (DG_T)_{11}(\zeta(s)) \end{bmatrix}
$$

Therefore, we have

$$
\begin{aligned}
&\int_{[0,1]} \left[\frac{\hat{\omega}(\zeta(s))}{\det(DG_T(\zeta(s)))} \cdot \frac{\hat{\mathbf{n}}(\zeta(s))}{\|(DG_T(\zeta(s)))^{-\top}\hat{\mathbf{n}}(\zeta(s))\|}\right]\hat{\eta}(s)\|DG_T(\zeta(s))\dot{\zeta}(s)\|ds \\
=&c\int_{[0,1]} [\hat{\omega}(\zeta(s)) \cdot \hat{\mathbf{n}}(\zeta(s))]\,\hat{\eta}(s)ds
\end{aligned}
$$

We conclude that (9.5) is equivalent with (9.8). This finishes the proof. $\square$

Proof of Lemma 9.10

Proof. By Lemma B.1, we have $\hat{\omega}(\hat{\mathbf{x}}) \in H(\hat{T})$, and

$$
\mathrm{div}\omega(\mathbf{x}(\hat{\mathbf{x}})) = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))}\mathrm{div}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}})
$$

for $\hat{\mathbf{x}} \in \hat{T}$ almost everywhere, provided we define $\hat{\omega}(\hat{\mathbf{x}})$ on $\hat{T}$ by

$$
\omega(\mathbf{x}(\hat{\mathbf{x}})) = \frac{DG_T(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}}))}\hat{\omega}(\hat{\mathbf{x}})
$$

for any $\hat{\mathbf{x}} \in \hat{T}$.

Using Definition 9.1, Definition 9.2, Lemma 9.4, Lemma 9.9, and Lemma 10 in [19], it is easy to see that

$$\Pi^2_{\tilde{r},T}\mathrm{div}\omega(\mathbf{x}(\hat{\mathbf{x}})) = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))}\Pi^2_{\tilde{r},\hat{T}}\mathrm{div}_{\hat{\mathbf{x}}}\hat{\omega}(\hat{\mathbf{x}}) = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))}\mathrm{div}_{\hat{\mathbf{x}}}\Pi^1_{\tilde{r}+1,\hat{T}}\hat{\omega}(\hat{\mathbf{x}})$$

$$\mathrm{div}\Pi^1_{\tilde{r}+1,T}\omega(\mathbf{x}(\hat{\mathbf{x}})) = \mathrm{div}\left[\frac{DG_T(\hat{\mathbf{x}})}{\det(DG_T(\hat{\mathbf{x}}))}\Pi^1_{\tilde{r}+1,\hat{T}}\hat{\omega}(\hat{\mathbf{x}})\right] = \frac{1}{\det(DG_T(\hat{\mathbf{x}}))}\mathrm{div}_{\hat{\mathbf{x}}}\Pi^1_{\tilde{r}+1,\hat{T}}\hat{\omega}(\hat{\mathbf{x}})$$

We have thus $\Pi^2_{\tilde{r},T}\mathrm{div}\omega = \mathrm{div}\Pi^1_{\tilde{r}+1,T}\omega$. □

Proof of Lemma 9.11

Proof. According to Lemma B.2 and Lemma B.3, there exist $\delta > 0$ and $C_1 > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$,

$$\|\Pi^1_{\tilde{r},T}\omega\|_{L^2(T)} \leq C_1\|\Pi^1_{\tilde{r},\hat{T}}\hat{\omega}\|_{L^2(\hat{T})} \quad \forall \omega \in L^2(T;\mathbb{R}^2)$$

$$\|\hat{\omega}\|^2_{H(\mathrm{div}_{\hat{\mathbf{x}}},T)} + \|\hat{\omega}\|^2_{L^2(\partial\hat{T})} \leq C_1\|\omega\|_{H^1(T)} \quad \forall \omega \in H^1(T;\mathbb{R}^2)$$

By Lemma 9.9, $\hat{\omega} \in H^1(\hat{T};\mathbb{R}^2)$ for any $\omega \in H^1(T;\mathbb{R}^2)$.

The definition of operator $\Pi_{\tilde{r},\hat{T}}$ implies that there exists a constant $C_2 > 0$ such that

$$\int_{\hat{T}}(\Pi^1_{\tilde{r},\hat{T}}\hat{\omega}(\hat{\mathbf{x}}))^\top\Pi^1_{\tilde{r},\hat{T}}\hat{\omega}(\hat{\mathbf{x}})d\hat{\mathbf{x}} \leq C_2(\|\hat{\omega}\|^2_{H(\mathrm{div}_{\hat{\mathbf{x}}},\hat{T})} + \|\hat{\omega}\|^2_{L^2(\partial(\hat{T}))}), \forall\hat{\omega}(\hat{\mathbf{x}}) \in [H^1(\hat{T})]^2.$$

It is easy to see that for affine meshes the above inequality holds for any $h > 0$. This finishes the proof. □

Proof of Theorem 9.15

Proof. For any $h > 0$ and any $T \in \mathcal{T}_h$, we define a linear isomorphism A_T from $H^1(\hat{T};\mathbb{R}^2)$ to $H^1(T;\mathbb{R}^2)$ by $(A_T\hat{\omega})(\mathbf{x}(\hat{\mathbf{x}})) = \dfrac{B_T\hat{\omega}(\hat{\mathbf{x}})}{\det(B_T)}$. It is easy to see that A_T is a linear isomorphism from $\mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$ to $\mathcal{P}_{\tilde{r}+2}\Lambda^0(T;\mathbb{R}^2)$.

We define an operator $E_T : H^1(\hat{T};\mathbb{R}^2) \longrightarrow \mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$ by $W_T(A_T\hat{\omega}) = A_T(E_T\hat{\omega})$. Obviously, W_T is well-defined if and only if E_T is well-defined. We denote by $\{\hat{\xi}_1, \cdots, \hat{\xi}_{l_{\tilde{r}}}\}$ a particular basis of $\mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T};\mathbb{R}^2)$.

According to Lemma B.4 and Lemma B.5, it is sufficient to show that there exist $\delta > 0$ and $C_1 > 0$ such that, for any $h < \delta$ and $T \in \mathcal{T}_h$, E_T is well-defined, and $\|(z_1, \cdots, z_{l_{\tilde{r}}})\| \leq C_1\|\hat{\omega}\|_{H^1(\hat{T})}$ for any $\hat{\omega}$. Here $\sum_{k=1}^{l_{\tilde{r}}} z_k\hat{\xi}_k = E_T\hat{\omega}$.

According to the definition of W_T, E_T can be defined by relations

$$\text{(C.1)} \qquad \int_T \mathrm{div}(A_TE_T\hat{\omega} - A_T\hat{\omega})(\mathbf{x})\hat{\psi}(\hat{\mathbf{x}}(\mathbf{x}))d\mathbf{x} = 0 \quad \forall\hat{\psi} \in \mathcal{P}_{\tilde{r}(T)}(\hat{T})/\mathbb{R}$$

(C.2)

$$\int_T((A_TE_T\hat{\omega})(\mathbf{x}) - (A_T\hat{\omega})(\mathbf{x}))^\top DG_T(\hat{\mathbf{x}}(\mathbf{x}))^{-\top}\hat{\mathbf{h}}_i(\hat{\mathbf{x}}(\mathbf{x}), t_{\tilde{r}(\hat{T})})d\mathbf{x} = 0 \quad 1 \leq i \leq k_{\tilde{r}}$$

(C.3)

$$\int_{[0,1]}[(A_TE_T\hat{\omega} - A_T\hat{\omega})(\mathbf{x}_e(s))\cdot\mathbf{n}(\mathbf{x}_e(s))]\hat{\eta}(s)\|\dot{\mathbf{x}}_e(s)\|ds = 0 \quad \forall\hat{\eta} \in \mathcal{P}_{\tilde{r}(e)}([0,1]), \forall e \in \triangle_1(T)$$

(C.4)

$$\int_{[0,1]}[(A_TE_T\hat{\omega} - A_T\hat{\omega})(\mathbf{x}_e(s))\cdot\mathbf{t}(\mathbf{x}_e(s))]\hat{\eta}(s)\|\dot{\mathbf{x}}_e(s)\|ds = 0 \quad \forall\hat{\eta} \in \mathcal{P}_{\tilde{r}(e)}([0,1]), \forall e \in \triangle_1(T)$$

$$E_T\hat{\omega} = 0 \text{ at all vertices of } \hat{T} \tag{C.5}$$

Denote:

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = B_T, J = \det(B_T), \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} = E_T\hat{\omega}, \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \end{bmatrix} = \hat{\omega}, \hat{u}_{i,j} = \frac{\partial \hat{u}_i}{\partial \hat{x}_j}, \hat{w}_{i,j} = \frac{\partial \hat{w}_i}{\partial \hat{x}_j}.$$

By pulling back to $\hat{T}$, E_T can be defined by relations

$$\begin{aligned}
&\int_{\hat{T}} J^{-1}[(b_{11}(DG_T)_{22} - b_{21}(DG_T)_{12})\hat{u}_{1,1} + (b_{12}(DG_T)_{22} - b_{22}(DG_T)_{12})\hat{u}_{2,1} \\
&\quad + (b_{21}(DG_T)_{11} - b_{11}(DG_T)_{21})\hat{u}_{1,2} + (b_{22}(DG_T)_{11} - b_{12}(DG_T)_{21})\hat{u}_{2,2}]\hat{\psi}(\hat{\mathbf{x}})d\hat{\mathbf{x}} \\
&= \int_{\hat{T}} J^{-1}[(b_{11}(DG_T)_{22} - b_{21}(DG_T)_{12})\hat{w}_{1,1} + (b_{12}(DG_T)_{22} - b_{22}(DG_T)_{12})\hat{w}_{2,1} \\
&\quad + (b_{21}(DG_T)_{11} - b_{11}(DG_T)_{21})\hat{w}_{1,2} + (b_{22}(DG_T)_{11} - b_{12}(DG_T)_{21})\hat{w}_{2,2}]\hat{\psi}(\hat{\mathbf{x}})d\hat{\mathbf{x}}, \\
&\quad \forall \hat{\psi} \in \mathcal{P}_{\tilde{r}(T)}(\hat{T})/\mathbb{R}
\end{aligned} \tag{C.6}$$

$$\begin{aligned}
&\int_{\hat{T}} (E_T\hat{\omega}(\hat{\mathbf{x}}))^\top B_T^\top DG_T(\hat{\mathbf{x}})^{-\top}\hat{\mathbf{h}}_i(\hat{\mathbf{x}}, t) \det(B_T^{-1}DG_T(\hat{\mathbf{x}}))d\hat{\mathbf{x}} \\
&= \int_{\hat{T}} (\hat{\omega}(\hat{\mathbf{x}}))^\top B_T^\top DG_T(\hat{\mathbf{x}})^{-\top}\hat{\mathbf{h}}_i(\hat{\mathbf{x}}, t) \det(B_T^{-1}DG_T(\hat{\mathbf{x}}))d\hat{\mathbf{x}} \quad 1 \le i \le k_{\tilde{r}}
\end{aligned} \tag{C.7}$$

$$\begin{aligned}
&\int_{[0,1]} [B_T E_T\hat{\omega}(\zeta(s)) \cdot (DG_T(\zeta(s))^{-\top}\hat{\mathbf{n}}(\zeta(s)))]\hat{\eta}(s) \det(B_T^{-1}DG_T(\zeta(s))ds \\
&= \int_{[0,1]} [B_T\hat{\omega}(\zeta(s)) \cdot (DG_T(\zeta(s))^{-\top}\hat{\mathbf{n}}(\zeta(s)))]\hat{\eta}(s) \det(B_T^{-1}DG_T(\zeta(s))ds \\
&\quad \forall \hat{\eta} \in \mathcal{P}_{\tilde{r}(e)}([0,1]), \forall e \in \triangle_1(T)
\end{aligned} \tag{C.8}$$

$$\begin{aligned}
&\int_{[0,1]} [B_T E_T\hat{\omega}(\zeta(s)) \cdot (DG_T(\zeta(s))\dot{\zeta}(s))]\hat{\eta}(s) \det(B_T^{-1})ds \\
&= \int_{[0,1]} [B_T\hat{\omega}(\zeta(s)) \cdot (DG_T(\zeta(s))\dot{\zeta}(s))]\hat{\eta}(s) \det(B_T^{-1})ds \quad \forall \hat{\eta} \in \mathcal{P}_{\tilde{r}(e)}([0,1]), \forall e \in \triangle_1(T)
\end{aligned} \tag{C.9}$$

$$E_T\hat{\omega} = 0 \text{ at all vertices of } \hat{T} \tag{C.10}$$

It is easy to see that (C.6) comes from (C.1), (C.7) comes from (C.2), (C.9) comes from (C.4), and (C.10) comes from (C.5). And (C.8) can be got from (C.3) by using the fact that $\|\dot{\mathbf{x}}_e(s)\| = \|DG_T(\zeta(s))\hat{\mathbf{t}}\| = c\|DG_T(\zeta(s))^{-\top}\hat{\mathbf{n}}\|$ for some non-zero constant c, which comes from direct calculation.

Notice that vector $\dot{\zeta}(s)$ is constant tangent vector along each edge of $\hat{T}$. Set

$$a = \hat{\mathbf{n}}^\top B_T^\top B_T\dot{\zeta}(s) \det(B_T^{-1}), \quad b = \frac{\det(B_T)\|\dot{\zeta}(s)\|}{\dot{\zeta}(s)^\top B_T^\top B_T\dot{\zeta}(s)}$$

Obviously, $b \neq 0$.

Perform now the operation: $b \times [(\text{C.9}) - a \times (\text{C.8})]$. We have,

$$\int_{[0,1]} [E_T \hat{\omega}(\zeta(s)) \cdot F_T(s)] \hat{\eta}(s) ds = \int_{[0,1]} [\hat{\omega}(\zeta(s)) \cdot F_T(s)] \hat{\eta}(s) ds \qquad \forall \hat{\eta} \in \mathcal{P}_{\tilde{r}(e)}([0,1]), \forall e \in \triangle_1(T) \tag{C.11}$$

where

$$F_T(s) = \det(B_T^{-1})[B_T^\top (DG_T(\zeta(s)))\dot{\zeta}(s) - \det(B_T^{-1} DG_T(\zeta(s)))(\hat{\mathbf{n}}^\top B_T^T B_T \dot{\zeta}(s)) B_T^\top (DG_T(\zeta(s)))^{-\top} \hat{\mathbf{n}})] \frac{\det(B_T)\|\dot{\zeta}(s)\|}{\dot{\zeta}(s)^\top B_T^\top B_T \dot{\zeta}(s)}$$

Then the definition of operator E_T can be rewritten by using conditions (C.6),(C.7), (C.8), (C.11), and (C.10).

Using the fact that $\hat{\mathbf{n}} \perp \dot{\zeta}(s)$, Lemmas 7.7,7.8,7.9, and the assumption that $(\mathcal{T}_h)_h$ is regular, we obtain

$$\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{e\in\triangle_1(T)} \sup_{s\in[0,1]} \|F_T(s) \cdot \frac{\dot{\zeta}(s)}{\|\dot{\zeta}(s)\|} - 1\| = \lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{e\in\triangle_1(T)} \sup_{s\in[0,1]} \|F_T(s) \cdot \hat{\mathbf{n}}\| = 0$$

Consequently,

$$\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{e\in\triangle_1(T)} \sup_{s\in[0,1]} \|F_T(s) - \hat{\mathbf{t}}\| = 0$$

We denote now by $E(T, \tilde{r})$ the matrix corresponding to the left-hand side of conditions (C.6),(C.7),(C.8), (C.11),(C.10), a particular basis of $\mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T}; \mathbb{R}^2)$ (the solution space), some basis of $\mathcal{P}_{\tilde{r}(T)}(\hat{T})/\mathbb{R}$, and some basis of $\mathcal{P}_{\tilde{r}(e)}([0,1])$ for each $e \in \triangle_1(T)$. We denote by $\{\hat{\xi}_1, \cdots, \hat{\xi}_{l_{\tilde{r}}}\}$ a basis for $\mathcal{P}_{\tilde{r}+2}\Lambda^0(\hat{T}; \mathbb{R}^2)$. Finally, we denote by $C(\tilde{r})$ the matrix corresponding to the left-hand side of conditions (5.14), (5.15),(5.16), and (5.17), and the same bases as above.

Using the fact that

$$\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \sup_{e\in\triangle_1(T)} \sup_{s\in[0,1]} \|F_T(s) - \hat{\mathbf{t}}\| = 0$$

and Lemmas 7.7,7.8, 7.9, we conclude that, for any $t \in [0,1]$,

$$\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \|E(T,\tilde{r}) - C(\tilde{r})\| = 0$$

Then, for any given $t \in [0,1]$, and any given $\tilde{r}$ with non-singular $C(\tilde{r})$, the matrix $E(T, \tilde{r})$ is non-singular for any $T \in \mathcal{T}_h$ when $h > 0$ small enough. Notice that the right-hand sides of conditions (C.6), (C.7),(C.8),(C.11), and (C.10) are continuous linear functionals of $\hat{\omega} \in H^1(\hat{T}; \mathbb{R}^2)$. We can conclude thus that the operator $W_{T,t}$ is well-defined for any $T \in \mathcal{T}_h$ with small enough h.

Since for any $t \in [0,1]$, $\lim_{h\to 0} \sup_{T\in\mathcal{T}_h} \|E(T,\tilde{r}) - C(\tilde{r})\| = 0$, and the matrix $C(\tilde{r})$ depends only on $\tilde{r}$, we can conclude that there exists $C_1 > 0$ such that when $h > 0$ small enough, then $\|(z_1, \cdots, z_{l_{\tilde{r}}})\| \leq C_1 \|\hat{\omega}\|_{H^1(\hat{T})}$ for any $T \in \mathcal{T}_h$. Here $\sum_{k=1}^{l_{\tilde{r}}} z_k \hat{\xi}_k = E_T \hat{\omega}$.

Finally, it is easy to see that the operator W_T will be well-defined and the inequality in the statement of this theorem holds for any $h > 0$ for affine meshes. This finishes the proof. ∎

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